Logic and the Methodology of Science
June 2004 Preliminary Exam

August 23, 2005

1.

(a) Outline a proof that the theory of rings is not decidable.

(b) Show that the set $\mathcal{V}$ of all valid formulae in the language of rings is not recursive.

2. Let $\mathcal{L}$ be the language with one binary relation symbol R, and no other nonlogical symbols. Let T be a consistent, decidable $\mathcal{L}$- theory.

(a) Show that there is a complete, consistent, decidable $\mathcal{L}$-theory U such that $T \subseteq U$.

(b) Show that T has a model $\mathcal{A} = (A, R^{\mathcal{A}})$ such that A and $R^{\mathcal{A}}$ are recursive.

3. Let M be a nonstandard model of Peano Arithmetic. Let $\theta(u, v)$ be the natural formula in the language of Peano arithmetic expressing " the u^{th} prime divides v". Show that for some $a \in M$,

$$\{i \in \omega \mid M \models \theta[\underline{i}^M, a]\}$$

is not recursive.

4. Let T be an axiomatizable $\mathcal{L}$-theory, and suppose there are only finitely many complete $\mathcal{L}$-theories U such that $T \subseteq U$. Show that T is decidable.

5. Let

$$A = \{e \mid W_e \text{ is infinite and } W_e \neq \omega\},$$

and

$$B = \{e \mid \varphi_e \text{ is total and range}(\varphi_e) \neq \omega\},$$

where φ_e is the e^{th} partial recursive function of one variable in some standard enumeration, and W_e is the domain of φ_e. Prove or refute: $A \leq_m B$.

6. Let T be a theory in a countable language, and suppose that T has an infinite model. Show that T has a model which admits a nontrivial automorphism.

7. Let $\mathsf{ZFC}^{\text{fin}}$ be ZFC, but with the axiom of infinity replaced by its negation, together with axioms defining names $\underline{a}$ for each herditarily finite set a. (If you prefer, you may do this problem with Peano Arithmetic replacing $\mathsf{ZFC}^{\text{fin}}$ throughout, letting $\underline{a}$ be the standard closed term for $a \in \omega$.) Let $\langle W_e \mid e \in \omega\rangle$ be a standard enumeration of the r.e. sets, as in problem 4 above. Let $\theta(u, v)$ be the natural formula in the language of $\mathsf{ZFC}^{\text{fin}}$ which formalizes the assertion "$W_u = W_v$".

(a) Show there is an e such that for all n, $\mathsf{ZFC}^{\text{fin}} + \theta(\underline{e}, \underline{n})$ is consistent. (Hint: use the recursion theorem.)

(b) Show that if e is as in part (a), then $W_e = \emptyset$.

(c) Show that there are e_0, e_1 such that for all m, n, $\mathsf{ZFC}^{\text{fin}} + \theta(\underline{e_0}, \underline{m}) + \theta(\underline{e_1}, \underline{n})$ is consistent.

(d) Use (c) to show that there are Π_1 sentences φ and ψ in the language of $\mathsf{ZFC}^{\text{fin}}$ such that

$$\mathsf{ZFC}^{\text{fin}} \nvdash \varphi \to \psi$$

and

$$\mathsf{ZFC}^{\text{fin}} \nvdash \psi \to \varphi.$$

8. A *graph* is a structure $\mathcal{G} = (V, E)$ for the language $\mathcal{L}$ with one binary relation symbol such that E is a symmetric relation on V. The $\mathcal{G}$-*component* of a vertex $v \in V$ is the set of all $u \in V$ such that there is a finite sequence

$\langle v_0, ..., v_n \rangle$ with $v = v_0, u = v_n$, and $v_i E v_{i+1}$ for all $i < n$. $\mathcal{G}$ is *connected* iff it has only one component.

Let $\mathcal{G}$ be a graph such that whenever $\varphi(u, v)$ is an $\mathcal{L}$-formula such that $\mathcal{G} \models \exists u \exists v \varphi(u, v)$, then there are a, b in the same component of $\mathcal{G}$ such that $\mathcal{G} \models \varphi[a, b]$. Show that there is a connected graph $\mathcal{H}$ such that $\mathcal{H} \equiv \mathcal{G}$.

9. An *existential* formula is one of the form $\exists v_0 ... \exists v_n \theta$, where θ is quantifier-free. An *AE* formula is one of the form $\forall v_0 ... \forall v_n \exists u_0 ... \exists u_k \theta$, where θ is quantifier-free. We write $\mathcal{A} \prec_1 \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$, and whenever $a_1, ..., a_n \in |\mathcal{A}|$ and φ is existential, then $\mathcal{A} \models \varphi[a_1, ..., a_n]$ iff $\mathcal{B} \models \varphi[a_1, ..., a_n]$.

(a) Suppose $\mathcal{A} \prec_1 \mathcal{B}$, and show there is a structure $\mathcal{C}$ such that $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{A} \prec \mathcal{C}$.

(b) Suppose that $\mathcal{A} \models \varphi$, for all AE sentences φ such that $T \vdash \varphi$. Show that there is a model $\mathcal{B} \models \mathcal{T}$ such that $\mathcal{A} \prec_1 \mathcal{B}$.

(c) Let T be a theory which is preserved under unions of substructure chains. Show that there is a set of axioms for T consisting of AE sentences.