GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE PRELIMINARY EXAMINATION

There are eight questions. Partial credit may be assigned for substantially correct partially worked solutions. To pass, you need a score of roughly fifty percent, though the ultimate decision about the passing mark will be decided by the committee of graders.

Throughout this exam, we use the standard notation φ_e to refer to the e^{th} partial recursive function relative to some standard listing and $W_e = \operatorname{dom} \varphi_e$ for the e^{th} recursively enumerable set.

1. Prove or disprove: there is a partial recursive $\psi(x)$ such that whenever W_e is finite, then $\psi(e)$ converges, and $|W_e| \leq \psi(e)$.

2. Let $\mathcal{L}$ be the first-order language generated by the signature having a single unary function symbol f. **Prove** that the empty $\mathcal{L}$-theory has a model companion. (In principle, this could be solved by abstract nonsense, but we would prefer to see an axiomatization of the model companion and then a proof that your axiomatization works.)

3. Let M be a model of PA and $\phi(x, y)$ a formula of $\mathcal{L}(+, \times, 0, 1)$. Let $c > 0$ be a positive element of M and let S be a subset of $\{a \in M : a < c\}$. Suppose that for all elements $a \in S$ one has $M \models (\exists x)\phi(x, a)$. **Show** that there is $b \in M$ such that for all $a \in S$, one has $M \models (\exists x < b)\phi(x, a)$.

4. Let $I := \{e \in \omega : W_e \text{ is infinite }\}$ and let $E := \{e \in \omega : W_e = \varnothing\}$. **Prove or disprove:** I recursive relative to E.

5. Let M be an $\mathcal{L}$-structure for some first order language $\mathcal{L}$. For each $i \in \omega$, let $A_i \preceq M$ be an elementary substructure. Let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ by countably many new unary predicate symbols P_i and let M' be the expansion to $\mathcal{L}'$ via the interpretation $P_i^{M'} = A_i$. We assume that for each finite set $F \subseteq \omega$ the set $\bigcap_{i \in F} A_i$ is an elementary substructure of M.

Show there is an elementary extension $N' \succeq M'$ for which there is an elementary substructure $B \preceq N := N' \upharpoonright \mathcal{L}$ (the reduct of N' to $\mathcal{L}$) such that

(i) for each $i \in \omega$, $B \subseteq P_i^{N'}$ and
(ii) $B \cap M = \bigcap_{i \in \omega} A_i$

6. For this problem you may take as given that every finite partial ordering can be extended to a linear ordering.

(i) If (A, R) is a partial ordering then there is a linear ordering $<$ of A such that $<$ extends R.
(ii) If (ω, R) is a recursive partial ordering then there is a Δ_2^0 set $S \subseteq \omega^2$ such that $R \subseteq S$ and (ω, S) is a linear ordering.

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7. Give an example of the following situation (and prove that your structure has the requisite properties).

- An $\mathcal{L}$-structure M for some first order language $\mathcal{L}$,
- elements $a \in M$ and $b \in M$,
- an elementary extension $N \succeq M$, and
- an automorphism $\sigma : N \to N$

such that $\sigma(a) = b$, but there is no automorphism $\tau : M \to M$ for which $\tau(a) = b$.

8. Let $M \models \text{PA}$.

- **Show** that there is no formula $\phi(x, y)$ such that for every definable set $D \subseteq M$ there is some parameter $b \in M$ for which $D = \{a \in M : M \models \phi(a, b)\}$.
- **Show** on the contrary that for any element $c \in M$ there is a formula $\vartheta(x, y)$ such that for any definable set $D \subseteq [0, c)^M := \{a \in M : 0 \leq a < c\}$ there is a parameter $d \in M$ with $D = \{a \in M : M \models \vartheta(a, d)\}$.