Preliminary Examination
Group in Logic and the Methodology of Science
September 21, 2020

Problem 1:

Suppose that T is a computable subtree of the tree of finite binary sequences.

1. Show that if T is infinite then T has an infinite path.
2. Show that if T has a unique infinite path then that path is computable.

Problem 2:

Consider the set A of indices for partial computable functions from the natural numbers to the rational numbers which have totally bounded range (i.e., whose range has a finite supremum and finite infimum). Show that A is Σ^0_2-complete.

Problem 3:

Show that PA is not finitely axiomitizable.

Problem 4:

Suppose that $A \subseteq \mathbb{N}$ is not computable. Show that there is another set B such that neither A nor B is computable relative to the other.

Problem 5:

Let $M_0 = (\mathbb{N}, s)$ (where s is the successor function) and let $M_0 \prec M_1$ be a proper elementary extension. Show that M_1 has a non-trival automorphism.

Problem 6: (a) Prove that if $\mathcal{M}$ and $\mathcal{N}$ are linear orderings, there is a linear ordering $\mathcal{C}$ such that $\mathcal{M}$ can be embedded into $\mathcal{C}$ and $\mathcal{N}$ can be elementary embedded into $\mathcal{C}$.

(b) Let $\mathcal{M} = (M, \leqslant)$ be an infinite structure, where $\leqslant$ defines a linear order. Assume that $Th(\mathcal{M})$ is model-complete. Show that the order $\leqslant$ is dense in M (that is, for any $a < b$ in M, there is $c \in M$ with $a < c < b$).

Problem 7:

Give an example of a structure M of size $\aleph_1$ such that $Th(M)$ is not ω-categorical, but any two countable elementary substructures of M are isomorphic.

Problem 8:

Recall that for any structure M and $A \subseteq M$, $\mathrm{dcl}(A)$ denotes the set of elements definable over A.

Let T be a theory such that for any model $M \models T$ and any $A \subseteq M$, we have $\mathrm{dcl}(A) \prec M$. Show that T has definable Skolem functions, that is: for every formula $\phi(x; \bar{y})$, there is a 0-definable function $f_\phi(\bar{y})$ such that

$$T \models (\forall \bar{y})\,((\exists x)\phi(x; \bar{y})) \longrightarrow \phi(f(\bar{y}); \bar{y}).$$