PRELIMINARY EXAMINATION
GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE

There are **eight** questions. Partial credit may be assigned for substantially correct solutions. The committee of examiners will determine what constitutes passing performance. You have three hours to complete this test. Please write your solutions on blank, loose sheets of paper.

1. Let $\mathcal{L}$ be a first-order language, T a (possibly incomplete) $\mathcal{L}$-theory, and Σ a set of $\mathcal{L}$-sentences. Suppose that for any pair of models $\mathfrak{A} \models T$ and $\mathfrak{B} \models T$, if $\mathfrak{A}$ and $\mathfrak{B}$ agree on Σ (that is, for every $\phi \in \Sigma$, $\mathfrak{A} \models \phi \iff \mathfrak{B} \models \phi$), then $\mathfrak{A} \equiv \mathfrak{B}$. **Show** for every $\mathcal{L}$-sentence ψ there is a sentence ϕ which is a finite Boolean combination of sentences from Σ for which $T \vdash (\psi \leftrightarrow \phi)$. (Note: in the case that $\Sigma = \varnothing$, we allow $\top$ and $\perp$ as Boolean combinations.)

2. Let $\mathcal{L} = \mathcal{L}(+, \cdot, <, 0, 1)$ be the language of arithmetic. Let T and U be two computably axiomatizable $\mathcal{L}$-theories. We suppose that $\mathrm{PA} \subseteq (T \cap U)$, $T \vdash \mathrm{Con}(U)$ and $U \vdash \mathrm{Con}(T)$. **Prove:** T is inconsistent.

3. Consider the structure $\mathfrak{R} := (\mathbb{R}, +, \{q\}_{q \in \mathbb{Q}})$ of the real numbers given with the usual addition operation and constant symbols for each rational number q interpreted in the obvious way. **Prove:** the set $\{(x, y) \in \mathbb{R}^2 \ : \ x < y\}$ is *not* definable in $\mathfrak{R}$.

4. Let $\langle W_e \rangle_{e \in \omega}$ be the standard enumeration of the computably enumerable subsets of $\omega \times \omega$. **Show** that the set

$$X := \{e \in \omega \ : \ W_e \text{ is an equivalence relation on } \omega \text{ having an infinite class } \}$$

is Σ^0_3-complete.

5. Let $\mathcal{L}$ be a first-order language. Recall that an $\mathcal{L}$-structure $\mathfrak{A}$ is *atomic* if for any finite tuple $a = (a_1, \ldots, a_n) \in A^n$ there is a formula $\phi(x_1, \ldots, x_n)$ for which $\mathfrak{A} \models \phi(a)$ and for any other formula $\vartheta(x_1, \ldots, x_n)$, either $\mathfrak{A} \models \phi \to \vartheta$ or $\mathfrak{A} \models \phi \to \neg\vartheta$. **Show** by giving an example with proof that it is possible to have nonisomorphic, elementarily equivalent atomic models $\mathfrak{A}$ and $\mathfrak{B}$ of the same *uncountable* cardinality. **Prove** (in detail) that if $\mathcal{L}$ is countable and $\mathfrak{A}$ and $\mathfrak{B}$ are countable elementarily equivalent atomic $\mathcal{L}$-structures, then $\mathfrak{A} \cong \mathfrak{B}$.

6. We say that a family S of subsets of ω is computably enumerable if there exists a computably enumerable set $R \subseteq \omega \times \omega$ so that

$$S = \{\{n \in \omega \ : \ \langle m, n \rangle \in R\} \ : \ m \in \omega\} \ .$$

Consider the family S of sets of the form $\{n\} \oplus X := \{2n\} \cup \{2m + 1 : m \in X\}$ where X is a computably enumerable set different from W_n (the n^{th} computably enumerable set). **Prove** that S is not computably enumerable.

7. Let $\mathcal{L}$ be a first-order language and let T be a consistent $\mathcal{L}$-theory which is axiomatized by universal sentences. Recall that a model $\mathfrak{A} \models T$ is *existentially closed* (in $\mathrm{Mod}(T)$) if for any formula $\phi(x) \in \mathcal{L}_A(x)$ in the one free variable x in the expansion of the language obtained by naming the elements of the universe of $\mathfrak{A}$, if there is some extension $\mathfrak{B} \supseteq \mathfrak{A}$ with $\mathfrak{B} \models T$ and $\mathfrak{B} \models (\exists x)\phi$, then $\mathfrak{A} \models (\exists x)\phi$. **Prove:** there are existentially closed models of T. **Show** by giving an example with proof that if we drop the hypothesis that T is universally axiomatized, then it may happen that T does not have any existentially closed models.

8. **Show** that there is a complete theory U in the language of arithmetic with $\mathrm{PA} \subseteq U$, $U \leq_T \varnothing''$, but $U \not\leq_T \varnothing'$.

Date: 28 August 2016.