Preliminary Examination
Group in Logic and the Methodology of Science
May 9, 2019

Problem 1:
Prove that the set of pairs $\{(e,i) \in \mathbb{N}^2 : W_e = \mathbb{N}\backslash W_i\}$ is Π^0_2 complete. (Write down this prove without using other known completeness results.)

Problem 2:
Let $\mathcal{L}$ be the first-order language having exactly two relation symbols: $<$ (which is binary) and E (which is binary). Let T be the $\mathcal{L}$ theory which is axiomatized by the following sentences.

- $<$ is a dense linear ordering without end-points.
- E is an equivalence relation with infinitely many equivalence classes.
- Each E-equivalence class is dense (i.e., $\forall x, y, z\ (x < y \to \exists w\ (E(z,w) \wedge x < w < y))$.)

(a) Show that T is consistent.
(b) Show that T is $\aleph_0$-categorical.

Problem 3:
Find infinitely many disjoint r.e. sets that are mutually recursively inseparable. That is, for any two of these sets, there is no recursive set that contains one and is disjoint from the other.

Problem 4:
Let M be an ω-saturated structure and let $f : M \to M$ be a function. Assume that there are formulas $\phi_i(x,y)$, $i < \omega$, with no parameters such that for any $(a,b) \in M^2$, we have

$$f(a) = b \iff M \models \bigwedge_{i<\omega} \phi_i(a,b).$$

Show that f is a definable function.

Problem 5:
(a) Show that there exists a model $\mathcal{M}$ of PA with an element m such that, for each $i \in \mathbb{N}$, the ith prime number divides m if and only if the ith sentence of arithmetic is true in $\mathcal{M}$.
(b) Show that m is not definable in $\mathcal{M}$ without parameters. (That is, that there is no formula $\varphi(x)$ such that $\mathcal{M} \models \exists! x\varphi(x)$ and $\mathcal{M} \models \varphi(m)$.)

Problem 6:
Let $(a_i : i < \omega)$ be an order-indiscernible sequence in a model $\mathcal{M}$. Let A, B be two disjoint subsets of ω and let $c \in \mathrm{acl}(a_A) \cap \mathrm{acl}(a_B)$ (where we use the notation a_A for $\{a_i : i \in A\}$). Recall that $\mathrm{acl}(\bar{b})$ denotes the model-theoretic algebraic closure of $\bar{b}$, that is the set of elements of $\mathcal{M}$ which satisfy by some algebraic formula $\varphi(x)$, where a formula is said to be *algebraic* if there are only of finitely many elements in $\mathcal{M}$ that make it true.
Prove that for any infinite set $C \subseteq \omega$, we have $c \in \mathrm{acl}(a_C)$.

Problem 7:
Let φ be a sentence of arithmetic such that $PA \vdash \varphi \to Con(PA)$. Use the fact that the 2nd Incompleteness theorem is provable in PA to show that

$$PA \vdash \varphi \leftrightarrow \big(\psi \wedge Con(PA + \psi)\big),$$

where ψ is the sentence $Con(PA) \to \varphi$.

Problem 8:
Let $\mathfrak{M}$ be the structure $(\mathbb{R}\backslash\{0\}; <)$. Show that the elements -1 and 1 have the same type but that there is no automorphism σ of $\mathfrak{M}$ with $\sigma(-1) = 1$.