

Modal Logic, on the cusp of philosophy and mathematics

roads not taken, shifting perspectives, and interdisciplinary influences

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Philosophical origins of modal logic

- 1 Aristotle's modal syllogistic, varieties of truth, ontological structure universe.
- 2 Frege's Begriffsschrift drops modalities, with an implicit rejection of Kant.
- 3 Lewis & Langford: strict implication, connected to modal logics.
Primary: proof intuitions. Striking: diversity of logical systems – cf. Bolzano's Program.
- 4 Meta-proof intuitions from ontology: H. B. Smith on the ontological ladder of modalities.

Mathematics enters

- 5 Poland. Algebraic semantics. Topological semantics, $\Box = \exists\forall$. Highlight: S4 the logic of IR.
Two main examples: metric spaces, trees. Motivations now include intuitionism.
- 6 Jónsson & Tarski, BAOs and representation in relational models.
Interplay algebra and model theory has persisted in modal logic until today.
- 7 Further mathematical sources of modalities: provability logic. Now $\Box\varphi$ is existential.

Confluence of streams

- 8 Kanger Hintikka Kripke: graph models, $\Box = \forall$. Analyze modal logics by frame constraints.
Special case of topological semantics: or rather, of neighborhood semantics.
- 9 Many more notions considered modal, and treated in this semantic paradigm:
time (Prior), knowledge, belief (Hintikka), deontic (Hansson), conditionals (Lewis), etc.
- 10 Influences from *linguistics*. Modality in natural language: high zoom level for reasoning.
Frege's 'eye versus microscope'. See Handbook of Philosophical Logic.
- 11 General approach: ordinary truth, modality ~ multiple reference, at points/tuples.
 $M, s \models \varphi$
- 12 Discussion: criticism of R , and of 'possible worlds'. Abstraction, take 'holistic view'.

Theory lines in the 1970s

- 13 *Translation*: not choose, but connect. Import classical model theory, Gestalt switch.
 $\Box\varphi : \forall y (Rxy \rightarrow [y/x]\varphi)$
Fragments of classical languages, modal logic as fine-structure, or 'zooming out'.
- 14 *Expressive power, invariance and simulation*. Erlanger Program.
Bisimulation invariance characterization of the modal language.
- 15 *Completeness theory*: still a major enterprise, start from logics – or from structures.
Beautiful examples: semantics provability logic, completeness Minkowski space-time.
- 16 Frame truth, monadic second-order logic. *Correspondence theory*, Sahlqvist Theorem.
- 17 Enter *universal algebra*. Goldblatt & Thomason theorem modally definable frame classes.
'The Three Pillars': Completeness Theory, Correspondence Theory, Algebraic Duality.

The era of applications in the 1980s

- 18 Computer science, economics, more mathematics, etc. Bring their own further structure.
 19 As well as general themes that enter the self-understanding of modal logic.
 From 'applied' to 'pure'. From decidability to computational complexity. Bisimulation and process invariance. 'The Balance' of expressive power and complexity in logic design.

Theory lines in the 1990s - 2010s

- 20 Propositional *language extensions*: Prior, Bulgaria, Hybrid Logic, = fine-structure FOL. Guarded Fragment, admit all quantified forms $\exists y(G(x, y) \ \& \ \varphi(x, y))$, and more. Open-ended. Most recent discovery: decidable 'Unary Negation Fragment'.
 21 Ongoing: *modal predicate logic*. How to combine modality, predication, quantifiers? Finding the best semantics is both a philosophical and a mathematical problem.
 22 Propositional dynamic logic, modal mu-calculus, *fixed-point logics* for induction and recursion. Connections with Automata Theory, Process Algebra, game semantics.
 23 Multi-agent systems. Knowledge, belief, action, games, preference.
Modal logic combination, surprisingly complicated. From trees to grids.
 24 More general frameworks. *Neighborhood semantics* revival.
 25 Return to the historical methodology. *Proof theory* for modal logics.
 26 All these research lines interconnect in the current literature.
 Richer languages for neighborhoods, Fix-point views of old modal logics.

Challenges

- 27 *Algebraic logic* today often ahead of modal model theory.
 28 Co-algebra: the rise of *category theory*.
 29 *Metaphysics*, hyperintensionality. Back to the lost philosophical paradise?
 Richer universes, notions of truth, hypergraphs.

What is modal logic?

- 30 Major turn in thinking: modality as exotic addition to classical logic or as fine-structure.
 31 Modal logic as 'classical logic plus operators' versus trend toward 'alternative logics'. Status of system translation. Many open problems even in the area of modal logics.
 32 Nothing in logic is what it seems. ML a fragment of FOL, or more general semantics?
 Modal foundations of predicate logic: Tarski semantics itself has a modal flavor.

$$\mathbf{M}, s \models \exists x\varphi \text{ iff for some } d \in D, \mathbf{M}, s[x:=d] \models \varphi$$

Leads to unusual decidable systems, and first-order extensions of first-order logic.

