On relating type theories to (intuitionistic) set theories

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Recent popularity of type theory

Scientific American, Quanta Magazine, Nautilus, ...

> *Voevodsky's Univalent Foundations require not just one inaccessible cardinal but an infinite string of cardinals, each inaccessible from its predecessor.*

Michael Harris, *Mathematics without apologies*, 2015.

Ian Hacking, *Why is there Philosophy of Mathematics at All?*, 2014.

Some "research" questions

- Take Martin-Löf type theory with all type constructors (**MLTT**), including W-types and infinitely many universes

$$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$$

- How strong is this theory?
- Not difficult to show that **ZFC** plus infinitely many inaccessibles is an upper bound.
- How strong is **MLTT** plus univalence for all universes?
- Now add the impredicative type **Prop** of propositions together with

$$\text{Prop} : \mathcal{U}_0$$

  How strong is this theory? (aka Calculus of inductive Constructions (CiC)).
- What are the set-theoretic counterparts (intuitionistic set theories) of such type theories?

Why Intuitionistic Theories?

- Philosophical Reasons: **Brouwer, Dummett, Martin-Löf, Feferman, Linnebo, ...**
- Computational content: Witness and program extraction from proofs.
- Intuitionistically proved theorems hold in more generality:
  The internal logic of most topoi is intuitionistic logic.
- Axiomatic Freedom Adopt axioms that are classically refutable but interesting and desirable.

Axiomatic Freedom or “New Worlds”

- May be it would be nice
- if all $f : \mathbb{N} \to \mathbb{N}$ were computable and those pesky non-standard models of **PA** didn’t exist?
- if all $f : \mathbb{R} \to \mathbb{R}$ were continuous and the world were Brouwerian?
- if all functions between manifolds were differentiable? (nilpotent non-zero infinitesimals)
- if there existed a set A with $\mathbb{N} \subseteq A$ such that A is in 1-1 correspondence with $A \to A$?
- if all $f : \mathbb{R} \to \mathbb{R}$ were measurable?
- if all homotopically equivalent sets could be viewed as identical (univalence)?

Type theory

- Types are structured collections of objects such as natural numbers.
- 1908 Russell:
  Mathematical logic as based on the theory of types
- 1910, 1912, 1913 Russell & Whitehead:
  Principia Mathematica
- 1926 Hilbert: *Über das Unendliche*
- 1940 Church: *A formulation of the simple theory of types*
- 1967 de Bruijn: *AUTOMATH*
- 1971 Martin-Löf: *A Theory of Types*

MLTT Judgements

A **judgement** has one of the following four forms:

- A type
  ("A is a well-formed type")
- $A = B$ type
  ("A and B are equal well-formed types")
- $a : A$
  ("a is a well-formed term of type A")
- $a = b : A$
  ("a and b are equal well-formed terms of type A")

Martin-Löf type theory as a deductive systems

One deduces sequents

$$\Gamma \vdash \mathfrak{A}$$

where Γ, called the **context**, is made up of variable declarations $(x : A)$ in the "right" order of dependency, and $\mathfrak{A}$ is a judgement.

The rules are divided into formation, introduction, elimination and equality rules.

The basic dependent type theory $\mathbf{MLTT}_{basic}$

$\mathbf{MLTT}_{basic}$ is the dependent type theory with the following forms of type:

- Bool, Empty and the type Nat of natural numbers.
- $\mathsf{List}(A)$, $A + B$ and $\mathsf{Id}(A, a, b)$.
- Dependent product: $\prod_{x:A} B(x)$
- Dependent sum: $\sum_{x:A} B(x)$

The Curry-Howard representation of the logical operations

- The standard approach of representing logic in Martin-Löf type theory is to view propositions (formulae, sentences) as types.
- The Σ type represents $\exists$.
- The Π type represents $\forall$.
- The $\times$ type represents $\wedge$.
- The $+$ type represents $\vee$.
- The $\rightarrow$ type represents $\supset$.
- Empty represents falsum.
- $\mathrm{Id}(A, a, b)$ to represent equality on A.

The full system **MLTT**

- has the W-type

$$\mathrm{W}_{x:A} B(x)$$

which is the type of *well-founded trees* over the family of types $(B(x))_{x:A}$.

W-types are a generalization of such types as natural numbers, lists, binary trees. They capture the "recursion" aspect of any inductive type.

- And it has infinitely many universes

$$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$$

- A universes is a type inhabited by types. Every universe is closed under all the previous type constructions and $\mathcal{U}_i : \mathcal{U}_{i+1}$.

Universes and Notation

- Universes $\mathcal{U}$ are types that contain types as elements.
- They contain Bool, Empty, Nat, and are closed under all the (other) type forming operations. E.g.

$$\frac{\Gamma \vdash A : \mathcal{U} \quad \Gamma, x : A \vdash B(x) : \mathcal{U}}{\Gamma \vdash (\prod_{x : A} B(x) : \mathcal{U}}$$

- Denote by $\mathbf{MLTT}^{-}$ the theory $\mathbf{MLTT}$ without W-types.
- $\mathbf{MLTT}_n$ is the subsystem with only n universes $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$.

  Furthermore, $\mathbf{MLTT}_n^{-}$ also lacks the W-type constructor.

Two Identities

- General equality rules (reflexivity, symmetry, transitivity) and substitution rules, simultaneously at the level of terms and types, apply to judgements. Re-write rules.

- But there is also propositional identity which gives rise to types $\mathsf{Id}(A, s, t)$ and allows for internal reasoning about identity.

  Shall write $s =_A t$ rather than $\mathsf{Id}(A, s, t)$

Higher identity structure on any type A

$$a =_A a'$$

$$p =_{a =_A a'} p'$$

$$\theta =_{p =_{a =_A a'} p'} \theta'$$

$$\vdots$$

In extensional type theory (Martin-Löf 1979, 1984) this hierarchy collapses, since $a =_A a'$ contains at most 1 element.

Not so in intensional type theory (Martin-Löf 1973, 1986). Groupoid model (Hofmann, Streicher 1994), Kan simplicial sets (Voevodsky 2010), Kan cubical sets (Bezem, Coquand, Huber 2013).

Extensional identity

$$(\text{Id–Formation}) \qquad \frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a =_A b \text{ type}}$$

$$(\text{Id–Introduction}) \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : a =_A a}$$

$$(\text{Id–Uniqueness}) \qquad \frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash p = 1_a : a =_A b}$$

$$(\text{Id–Reflection}) \qquad \frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash a = b : A}.$$

- Reflection makes judgemental identity undecidable, i.e., the (type checking) questions whether $\Gamma \vdash a = b : A$ or $\Gamma \vdash a : A$ hold become undecidable.

New identity laws, Martin-Löf 1973

Indiscernability of Identicals:

If $p : a =_A b$ and $P(a)$ then $P(b)$.

This entails a transport function $t(p) : P(a) \to P(b)$.

Generalization: Now suppose that

$$d(x) : C(x, x, 1_x)$$

holds for all $x : A$.

Then d can be extended to a function $\tilde{\mathrm{J}}_d$ on

$$\sum_{x,y\,:\,A} x =_A y$$

i.e., if $a, b : A$ and $p : a =_A b$ then

$$\begin{aligned} \tilde{\mathrm{J}}_d(a, b, p) \quad &: \quad C(a, b, p) \\ d(a) = \tilde{\mathrm{J}}_d(a, a, 1_a) \quad &: \quad C(a, a, 1_a) \end{aligned}$$

Rules for intensional identity

$$
\text{(Id–Elim)} \qquad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \\ \Gamma \vdash p : a =_A b \\ \Gamma, x : A, y : A, z : x =_A y \vdash C(x, y, z) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathrm{J}(d, a, b, p) : C(a, b, p)}
$$

$$
\text{(Id–Eq)} \qquad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma, x : A, y : A, p : x =_A y \vdash C(x, y, p) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathrm{J}(d, a, a, 1_a) = d(a) : C(a, a, 1_a)}
$$

Strengths of **MLTT**?

- 1980s work on Martin-Löf type theory by Aczel, Beeson, Feferman, Hancock, Jervell, ...
- Early 1990's: proof-theoretic tools were in place to determine the exact strength of Martin-Löf type theories with finitely many universes, infinitely many universes, W-types, no W-types, super univ., Mahlo-univ., etc.
- E. Palmgren (1992)
- R. (1993)
- A. Setzer (1998)

Myhill's Constructive set theory 1975

CST based on intuitionistic logic

Many sorted system: **numbers, sets, functions**

Axioms (simplified)

- **Extensionality**
- **Pairing**, **Union**, **Infinity** (or $\mathbb{N}$ is a set)
- **Bounded Separation**
- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.
- **Replacement**
- **Set Induction Scheme**

Moving between type theory and set theory

The types-as-sets interpretation (TaS).

$$\text{type theory} \hookrightarrow \text{set theory}$$

Aczel (late 1970's): The sets-as-trees interpretation (SaT)

$$\text{set theory} \hookrightarrow \text{type theory}$$

Constructive Zermelo-Fraenkel set theory, **CZF**

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Bounded Separation**
- **Subset Collection**

  For all sets A, B there exists a "sufficiently large" set of multi-valued functions from A to B.

- **Strong Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x,y) \ \rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\ \varphi(x,y) \ \wedge\ (\forall y \in b)\, (\exists x \in a)\ \varphi(x,y)\,]$$

- **Set Induction scheme**

Three notions of large set

- A set A is said to be **regular** if it is inhabited and transitive and whenever $B \in A$ and R is a set relation such that $\forall x \in B \, \exists y \in A \; R(x, y)$ then there exists $C \in A$ such that $\forall x \in B \, \exists y \in C \; R(x, y)$ and $\forall y \in C \, \exists x \in B \; R(x, y)$.
- Denote by $\mathbf{CZF}^-$ the theory $\mathbf{CZF}$ without the Set Induction scheme.
- A set I is said to be **weakly inaccessible** if I is a regular set such that $I \models \mathbf{CZF}^-$.
- A set I will be called **inaccessible** if I is weakly inaccessible and for all $x \in I$ there exists a regular set $y \in I$ such that $x \in y$.

An 'algebraic' characterization of "inaccessibility"

Proposition ($\mathbf{CZF}^-$)

A set I is weakly inaccessible iff I is a regular set such that the following are satisfied:

1. $\omega \in I$,
2. $\forall a \in I \ \bigcup a \in I$,
3. $\forall a \in I \, [a \text{ inhabited } \Rightarrow \bigcap a \in I]$,
4. $\forall A, B \in I \, \exists C \in I \quad C$ is full in $\mathbf{mv}({}^{A}B)$.

How strong is $\mathbf{MLTT}^-$ plus Univalence?

Recall that $\mathbf{CZF}^-$ denotes the theory $\mathbf{CZF}$ without the Set Induction scheme.

Theorem 1. (Crosilla, R. 2002)

The theory

$$\mathbf{CZF}^- + \forall x\, \exists I\, [x \in I \;\wedge\; I \text{ weakly inaccessible}]$$

has the same strength as

$$\mathbf{ATR}_0$$

so has proof-theoretic ordinal Γ_0.

Proposition. $\mathbf{MLTT}^-$ can be interpreted in

$$\mathbf{CZF} + \text{weak-INACC}$$

where weak-INACC stands for $\forall x\, \exists I\, [x \in I \;\wedge\; I \text{ weakly inaccessible}]$.

Theorem 2. $\mathbf{MLTT}^- + \mathrm{UA}$ can be interpreted in $\mathbf{CZF} + \text{weak-INACC}$, too.

Here UA asserts that all universes are univalent.

The Bezem-Coquand-Huber constructive Kan cubical sets model can be done in this theory.

Corollary. All the theories $\mathbf{MLTT}^-$, $\mathbf{CZF} + \text{weak-INACC}$, and $\mathbf{MLTT}^- + \mathbf{UA}$ are of the same strength.

It does not matter whether the identity type is extensional or intensional.

It was known by work of Jervell 1978 and Feferman 1980 that (extensional) $\mathbf{MLTT}^-$ has strength Γ_0.

Univalence

- Let $f, g : \prod_{x:A} P(x)$. A **homotopy** from f to g is a dependent function of type
$$(f \simeq g) :\equiv \prod_{x:A} (f(x) =_{P(x)} g(x)).$$
- Let $f : A \vdash B$.
$$\mathsf{isequiv}(f) :\equiv \Big(\sum_{g:B\to A} (f \circ g \simeq \mathsf{id}_B)\Big) \times \Big(\sum_{h:B\to A} (h \circ f \simeq \mathsf{id}_A)\Big).$$
- $(A \simeq B) :\equiv \sum_{f:A\to B} \mathsf{isequiv}(f)$.
- For types $A, B : \mathcal{U}$ there is a canonical function
$$\mathsf{idtoeqv} : (A =_{\mathcal{U}} B) \vdash (A \simeq B).$$
The **Univalence Axiom** asserts that this function is itself an equivalence:
$$(A =_{\mathcal{U}} B) \simeq (A \simeq B).$$

Strength of **MLTT**

Theorem:
The following theories prove the same arithmetical statements:

(i) **MLTT**.

(ii) *The extensional type theory* $\mathbf{MLTT}^{ext}$.

(iii) **CZF** *plus for every* $n \in \mathbb{N}$, *an axiom asserting that there is a tower of* n*-many inaccessible sets,* $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n$.

(iv) $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n + \mathrm{RDC} +$ *Presentation Ax,*

where RDC *signifies the* relativized dependent choices axiom.

“Classical” Strength of **MLTT**

- It’s the same as

$$\mathbf{KP} + \{n\text{-many recursively inaccessible ordinals}\}_{n \in \mathbb{N}}$$

or

$$\Delta^1_2\text{-CA} + \{n \text{ tower of } \beta\text{-models of } \Delta^1_2\text{-CA}\}_{n \in \mathbb{N}}$$

- The strength of all of these theories is considerable but tiny when compared to Π^1_2-$\mathbf{CA}_0$.
- Does the addition of the Univalence Axiom change that picture?
- No, since the cubical model of **Bezem, Coquand, Huber** can be done “constructively” in type theory, though not all types have been included yet.

  For details see M. Rathjen *Proof Theory of Constructive Systems: Inductive Types and Univalence*, arXiv:1610.02191 (2016). To appear in: “Feferman on Foundations Logic, Mathematics, Philosophy”.

Vicious circles

"... vicious circles ... [arise] from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. [....] We shall, therefore, have to say that statements about 'all propositions' are meaningless. By saying that a set has 'no total,' we mean, primarily, that no significant statement can be made about 'all its members.' In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting."
Whitehead & Russell

- So we must be very careful about introducing the notion of proposition.
- There are predicative approaches to this which lead to level restrictions as in Principia and allow only "smaller collections" into which **Prop** is broken, such as Martin-Löf's **universes**.
- Or one sticks to the impredicative approach but restricts the type forming operations in other ways as for instance done in system **F**.

We shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this, that every propositional function is equivalent, for all its values, to some predicative function of the same arguments. [...] We will call this assumption the axiom of classes, or the axiom of reducibility.

The Russell-Prawitz interpretation of logic

- Papers by Russell from 1903 and 1906 contain the idea of possible definitions of
$$\wedge, \vee, \neg, \exists$$
in terms of $\rightarrow$ and $\forall$ via quantification over propositions $\forall p$:
$$\begin{aligned}
\varphi \wedge \psi &\equiv \forall p[(\varphi \rightarrow (\psi \rightarrow p)) \rightarrow p] \\
\varphi \vee \psi &\equiv \forall p[(\varphi \rightarrow p) \rightarrow ((\psi \rightarrow p) \rightarrow p)] \\
\neg\varphi &\equiv \forall p[\varphi \rightarrow p] \\
\exists x \varphi(x) &\equiv \forall p[\forall x(\varphi(x) \rightarrow p) \rightarrow p]
\end{aligned}$$
- Prawitz showed in (1965) that the above equivalences hold in second order intuitionist logic.

The type Prop

- In fact, the above equivalences can be used as definitions in the $\rightarrow, \forall$ fragment of second order intuitionistic logic, thereby reducing full intuitionistic second order logic to this fragment.
- This idea is also used to express logic in Girard's system **F** (1971) and is the standard approach to representing logic in the **calculus of constructions** (Coquand 1990) and extensions.
- The standard approach to representing logic in the type theory **Lego** (Luo & Pollack 1992; Luo 1994) and also, sometimes, the type theory of **Coq** (Barras et al. 1996), is to use the above **Russell-Prawitz** representation, where the variable p ranges over the the impredicative type called

  Prop

The central rule for **Prop**

- Propositions are represented as objects of type **Prop**. These objects are themselves types (or names of types in the Tarski treatment).

$$\frac{\Gamma, x : A \ \vdash \ B(x) : \mathbf{Prop}}{\Gamma \ \vdash \ \prod_{x:A} B(x) : \mathbf{Prop}}$$

- Note that this rule is highly impredicative as A can be any type (e.g. **Prop**).

The type Prop in more detail

$$\mathrm{Prop} : \mathcal{U}_0 \qquad \mathrm{Empty} : \mathrm{Prop} \qquad \frac{A : \mathrm{Prop}}{A \text{ type}}$$

$$\frac{A \text{ type} \quad x : A \vdash B(x) : \mathrm{Prop}}{\prod_{x : A} B(x) : \mathrm{Prop}}$$

$$\frac{A : \mathrm{Prop} \quad b_1 : A \quad b_2 : A}{b_1 = b_2 : A}$$

How strong is **MLTT** + Prop?

- **ZFC** plus infinitely many inaccessible cardinals suffices.
- Seems to be a difficult problem.
- Let's treat restricted cases first.

The type Prop reflecting just $\mathcal{U}_0$

$$\mathrm{Prop} : \mathcal{U}_0 \qquad \mathrm{Empty} : \mathrm{Prop} \qquad \frac{A : \mathrm{Prop}}{A : \mathcal{U}_0}$$

$$\frac{A : \mathcal{U}_0 \quad x : A \vdash B(x) : \mathrm{Prop}}{\prod_{x : A} B(x) : \mathrm{Prop}}$$

$$\frac{A : \mathrm{Prop} \quad b_1 : A \quad b_2 : A}{b_1 = b_2 : A}$$

Prop embodies Powerset

Theorem. The following theories have the same proof-theoretic strength

(i) $\mathbf{MLTT_1V}$ + Prop reflecting types in $\mathcal{U}_0$.

(i) Power Kripke-Platek set theory, $\mathbf{KP}(\mathcal{P})$

(ii) $\mathbf{CZF} + \mathrm{Powerset}$

Now let's stick to one universe but strengthen the rules for Prop so that it reflects all types A.

$$\frac{A : \mathrm{type} \quad x : A \vdash B : \mathrm{Prop}}{\prod_{x : A} B(x) : \mathrm{Prop}}$$

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Full Separation**
- **Powerset**
- **Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x,y) \quad \vdash \quad \exists b\ (\forall x \in a)\, (\exists y \in b)\ \varphi(x,y)$$

- **Set Induction**

$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x) \rightarrow\ \varphi(a)) \rightarrow \forall a\ \varphi(a),$$

- **IZF** has the same strength as **ZF** (Friedman).

Two set-theoretic axioms pertaining to **Prop**

- Gambino 2000
- **Negative Separation**

$$\exists y\, \forall x\, [x \in y \;\leftrightarrow\; x \in a \,\wedge\, \neg\neg\varphi(x)]$$

  for all formulae $\varphi(x)$.

- **Negative Power Set**

$$\exists z\, \forall x\, [x \in z \;\leftrightarrow\; x \subseteq a \,\wedge\, \forall u \in a\, (\neg\neg u \in x \vdash u \in x)]$$

Negative Intuitionistic Zermelo-Fraenkel set theory, $\mathbf{IZF}^{\neg\neg}$

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Bounded Separation**
- **Negative Separation**
- **Subset Collection**
- **Negative Powerset**
- **Strong Collection**
- **Set Induction**

A model of **IZF**$^{\neg\neg}$ which is not a model of **IZF**

- Andrew Swan (2012)
- Class realizability over $V(\mathcal{A})$ where $\mathcal{A}$ is a class order pca.
- Works for all axioms of **CZF** except maybe Bounded Separation.
- If $\mathcal{A}$ satisfies an extra condition, dubbed *uniformity* by Swan, then also $V(\mathcal{A}) \models$ Bounded Separation.
- Let $\Lambda(V)$ be the λ-terms over V. Let $\mathcal{T}$ be the set of equivalence classes modulo β-reduction.
  Then $V(\mathcal{T}) \models$ **CZF** but refutes Powerset.
- $V(\mathcal{T}) \models$ negative Powerset + negative Separation.

The strength of $\mathbf{IZF}^{\neg\neg}$

Theorem: $\mathbf{IZF}^{\neg\neg}$ is of the same strength as $\mathbf{MLTT}_1\mathbf{V} + \mathrm{Prop}$

Conjecture: $\mathbf{IZF}^{\neg\neg}$ is much weaker in strength than **ZFC**.

Impredicative Moves in HoTT

$\mathsf{isProp}(P) \ := \ \prod_{x,y:P} x =_P y$

$\mathsf{Prop}_{\mathcal{U}} \ := \ \{A : \mathcal{U} \mid \mathsf{isProp}(A)\}.$

Axiom of Propositional Resizing

$$\mathsf{Prop}_{\mathcal{U}_i} \ \to \ \mathsf{Prop}_{\mathcal{U}_{i+1}}$$

is an equivalence.

$\Omega := \mathsf{Prop}_{\mathcal{U}_0}$

$\mathcal{P}(A) \ := \ (A \to \Omega).$

Proof Theory & Constructivism

Strands

1. Proof-theoretic strength of theories (ordinal analysis, reduction, (partial) conservativity, speed up, classifications of provable functions, phase transitions, combinatorial independence results)
2. Proofs as (mathematical objects) (structural proof theory): cut elimination, normalization, ...
3. Proof interpretations (functional, realizability, negative and A-translations, ...)
4. Extraction of additional information from proofs (proof mining) (computational, constructive, bounds, uniformities, ...).
5. Proof complexity, mostly propositional, bounded arithmetic, P vs NP
6. Computer-based: Verification of proofs (proof assistants), search/construction of proofs
7. Constructive Mathematics
8. Intuitionistic "worlds" (topos theory, realizability/sheaf/Heyting-valued models, ...).
9. Type theories, λ-calculi, ...

Proof Theory & Constructivism

Underdeveloped areas/connections

1. Proof theory/constructivism and philosophy
2. Ordinal analysis of strong theories (set theory etc.)
3. Intuitionistic set theory (also with very large set axioms).
4. Proof theory and computability (higher types)
5. Proof theory of infinitary logics (both classical & intuitionistic)
6. Identity of proofs
7. Understanding/representing non-formal proofs: geometry, pictures, analogy, ...