Computability Theory: A Jaunt

Denis R. Hirschfeldt — University of Chicago

Logic at UC Berkeley, May 5th, 2017

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"Computability is perhaps the most significant and distinctive notion modern logic has introduced. . . "

— Wilfried Sieg *

* "On computability", *Handbook of the Philosophy of Mathematics*, 2009

Disclaimers

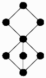
Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut enim, inquit, gubernator aeque peccat, si palearum navem evertit et si auri, item aeque peccat, qui parentem et qui servum iniuria verberat. Qui si omnes veri erunt, ut Epicuri ratio docet, tum denique poterit aliquid cognosci et percipi. Si ista mala sunt, in quae potest incidere sapiens, sapientem esse non esse ad beate vivendum satis. Non metuet autem, sive celare poterit, sive opibus magnis quicquid fecerit optinere, certeque malet existimari bonus vir, ut non sit, quam esse, ut non putetur. Unum, cum in voluptate sumus, alterum, cum in dolore, tertium hoc, in quo nunc equidem sum, credo item vos, nec in dolore nec in voluptate; Quibus natura iure responderit non esse verum aliunde finem beate vivendi, a se principia rei gerendae peti; Istud quidem, inquam, optime dicis, sed quaero nonne tibi faciendum idem sit nihil dicenti bonum, quod non rectum honestumque sit, reliquarum rerum discrimen omne tollenti. Duo Reges: constructio interrete.

Eaedem enim utilitates poterunt eas labefactare atque pervertere. Si enim Zenoni licuit, cum rem aliquam invenisset inusitatam, inauditum quoque ei rei nomen inponere, cur non liceat Catoni? Quasi vero aut concedatur in omnibus stultis aeque magna esse vitia, et eadem inbecillitate et inconstantia L. Quoniamque non dubium est quin in iis, quae media dicimus, sit aliud sumendum, aliud reiciendum, quicquid ita fit aut dicitur, omne officio continetur. Pomponius Luciusque Cicero, frater noster cognatione patruelis, amore germanus, constituimus inter nos ut ambulationem postmeridianam conficeremus in Academia, maxime quod is locus ab omni turba id temporis vacuus esset. Ego autem tibi, Piso, assentior usu hoc venire, ut acrius aliquanto et attentius de claris viris locorum admonitu cogitemus. Cur igitur easdem res, inquam, Peripateticis dicentibus verbum nullum est, quod non intellegatur? Quoniam igitur, ut medicina valitudinis, navigationis gubernatio, sic vivendi ars est prudente, necesse est eam quoque ab aliqua re esse constitutam et profectam. Nunc reliqua videamus, nisi aut ad haec, Cato, dicere aliquid vis aut nos iam longiores sumus. Atque haec contra Aristippum, qui eam voluptatem non modo summam, sed solam etiam ducit, quam omnes unam appellamus voluptatem. Teneamus enim illud necesse est, cum consequens aliquod falsum sit, illud, cuius id consequens sit, non posse esse verum. Qui autem diffidet perpetuitati bonorum suorum, timeat necesse est, ne aliquando amissis illis sit miser. Inest in eadem explicatione naturae insatiabilis quaedam e cognoscendis rebus voluptas,in qua una confectis rebus necessariis vacui negotiis honeste ac liberaliter possimus vivere. Non ergo Epicurus ineruditus, sed ii indocti, qui, quae pueros non didicisse turpe est, ea putant usque ad senectutem esse discenda. Ratio quidem vestra sic cogit. Vos autem cum perspicuis dubia debeatis illustrare, dubiis perspicua conamini tollere. Ego autem: Ne tu, inquam, Cato, ista exposuisti, ut tam multa memoriter, ut tam obscura, dilucide, itaque aut omittamus contra omnino velle aliquid aut spatium sumamus ad cogitandum;

Tantus est igitur innatus in nobis cognitionis amor et scientiae, ut nemo dubitare possit quin ad eas res hominum natura nullo emolumento invitata rapiatur. Itaque illa non dico me expetere, sed legere, nec optare, sed sumere, contraria autem non fugere, sed quasi secernere. Vos autem cum perspicuis dubia debeatis illustrare, dubiis perspicua conamini tollere. Quid enim mihi potest esse optatius quam cum Catone, omnium virtutum auctore, de virtutibus disputare? In omni enim arte vel studio vel quavis scientia vel in ipsa virtute optimum quidque rarissimum est. Utrum enim sit voluptas in iis rebus, quas primas secundum naturam esse diximus, necne sit ad id, quod agimus, nihil interest. Haec non erant eius, qui innumerabilis mundos infinitasque regiones, quarum nulla esset ora, nulla extremitas, mente

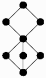

Andante (♩ = 63)
legato e sostenuto
ten.
p
molto espress.
pp
cresc.
ten.
dim.

A few themes



Reducibilities and degree structures

A few themes



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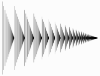
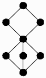


Nonstandard and uncountable settings

A few themes



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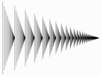

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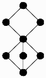
Teratology and monstricide

A few themes



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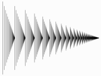

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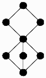

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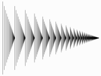

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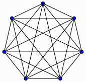
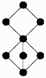


Interactions with other fields

A few themes



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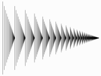

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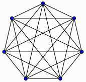

Interactions with other fields

Berkeley

"***Recursion Theory:*** *That part of mathematical logic which is focused on definability, especially for subsets of the natural numbers (ω) and of the real numbers (2^ω).*"

— *Ted Slaman* *

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$\mathcal{C} \subseteq 2^\omega$ is Π^0_1 iff there is a computable binary tree whose infinite paths are the elements of $\mathcal{C}$.

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An infinite set C is cohesive for $R_0, R_1, \dots$ if $\forall i\,(C \subseteq^* R_i \,\vee\, C \subseteq^* \overline{R_i})$.

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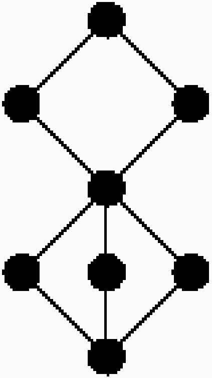
Thm (Jockusch and Stephan). An oracle can find cohesive sets for all uniformly computable sequences iff its jump has PA Turing degree over $\mathbf{0}'$.



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An e-degree is total if it is in the image of this embedding.

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Thm (Cai, Ganchev, Lempp, Miller, and Soskova). The total e-degrees are definable in the e-degrees.

For a set S, let $\rho_n(S) = \frac{S \cap \{0,1,\ldots,n-1\}}{n}$.

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A generic description of A is a partial function f s.t. $\underline{\rho}(\mathrm{dom}(f)) = 1$ and $f(n) = A(n)$ where defined.

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Igusa has significant partial results.

A is coarsely computable at density r if there is a computable set C such that $\underline{\rho}(\{n : C(n) = A(n)\}) \geqslant r$.

Let $\gamma(A) = \sup\{r : A \text{ is coarsely computable at density } r\}$.

Coarse computability bounds

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If $\mathbf{a}$ is a hyperimmune Turing degree then $\Gamma(\mathbf{a}) = 0$.

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Thm (Monin). $\Gamma(\mathbf{a})$ and $\Gamma_{\mathtt{tt}}(\mathbf{a})$ are always 0, $\frac{1}{2}$, or 1.

Thm (Martin) (AD). If $\mathcal{C} \subseteq 2^\omega$ is closed under $\equiv_{\mathbf{T}}$ then either $\mathcal{C}$ or $\overline{\mathcal{C}}$ contains a cone $\{X : A \leqslant_{\mathbf{T}} X\}$.

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The degree spectrum of a relation R on a structure $\mathfrak{A}$ is $\mathrm{DgSp}_{\mathfrak{A}}(R) = \{\mathrm{deg}_{\mathrm{T}}(R^{\mathfrak{B}}) : \mathfrak{B}$ is a computable copy of $\mathfrak{A}\}$.

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Open Question. Can these results be extended beyond 2?

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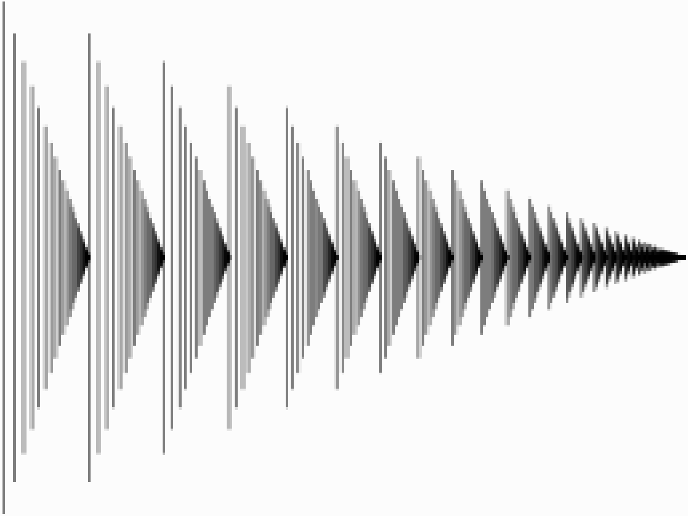
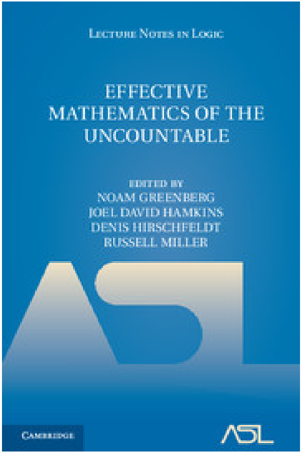
Thm (Montalbán) (PD). T is a counterexample to Vaught's Conjecture iff the class of countable models of T is uncountable (up to isomorphism) and satisfies hyperarithmetic-is-recursive on a cone.

Part III: Effective Mathematics of the Uncountable



Shameless Plug I



Approaches discussed in the book:

- $\mathbb{R}$-computability (Calvert and Porter)
- Infinite time Turing machines (Coskey and Hamkins)
- Admissible computability on ω_1 (Greenberg and Knight)
- Local computability (Miller)
- Borel structures (Montalbán and Nies)
- E-recursion (Sacks)
- Reverse mathematics (Shore)
- Σ-definability (Stukachev)

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Schweber extended this idea to possibly uncountable structures:

$\mathfrak{A}$ is generically Muchnik reducible to $\mathfrak{B}$ if for any generic extension $V[G]$ in which $\mathfrak{A}$ and $\mathfrak{B}$ are countable, $V[G] \vDash \mathfrak{A} \leqslant_{\mathrm{w}} \mathfrak{B}$.

$\mathcal{C} \subseteq 2^\omega$ is Muchnik (or weakly) reducible to $\mathcal{D} \subseteq 2^\omega$, written $\mathcal{C} \leqslant_{\mathrm{w}} \mathcal{D}$, if every element of $\mathcal{D}$ computes an element of $\mathcal{C}$.

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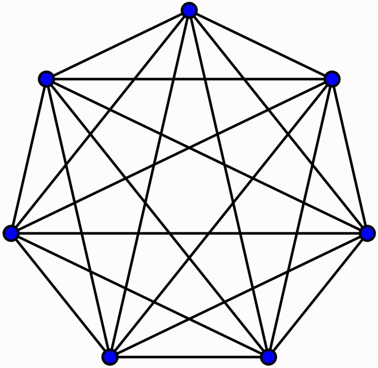
Knight, Montalbán, and Schweber showed that it does not matter if we take “any” to mean “there exists” or “for all”.



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Computability, Analysis, and Geometry, Banff, 2015

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A monotonic $f : [0, 1] \to \mathbb{R}$ is differentiable almost everywhere.

Thm (Brattka, Miller, and Nies). $x \in [0, 1)$ is computably random iff every computable monotonic $f : [0, 1] \to \mathbb{R}$ is differentiable at x.

A dynamical system consists of a probability space $(\Omega, \mathcal{B}, \mu)$ and a function $T : \Omega \to \Omega$ s.t. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$.

Thm (Birkhoff). If $f : \Omega \to \mathbb{R}$ is L^1 then

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

exists for almost all x.

We call such x weak Birkhoff for $(\Omega, \mathcal{B}, \mu, T)$ and f.

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We call such x weak Birkhoff for $(\Omega, \mathcal{B}, \mu, T)$ and f.

Fix $\Omega = 2^\omega$ with the uniform measure μ.

Thm (V'yugin / Franklin and Towsner). $A \in 2^\omega$ is Martin-Löf random iff A is weak Birkhoff for all computable T and f.

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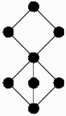
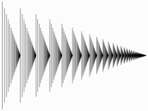
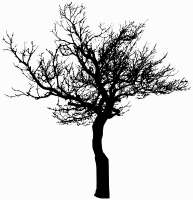
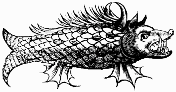
Thm (N. Lutz). This Intersection Formula holds for *all* $\mathcal{C}, \mathcal{D} \subseteq \mathbb{R}^n$.

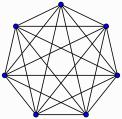
Part V: Reverse mathematics and computability of combinatorics



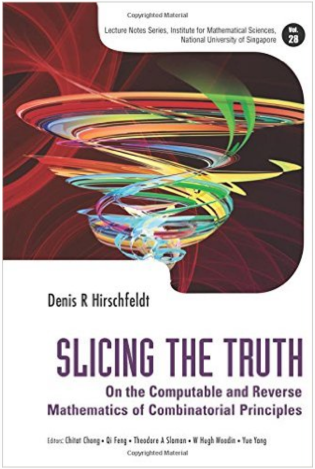








Shameless Plug II



We work in a two-sorted 1st order language with number variables, set variables, and symbols $0, 1, S, <, +, \cdot, \in$.

A model in this language consists of a 1st order part $\mathcal{N} = (N; 0_N, 1_N, S_N, <_N, +, \cdot_N)$ and a 2nd order part $\mathcal{S} \subseteq 2^N$.

If $\mathcal{N}$ is standard, we call this an ω-model and identify it with $\mathcal{S}$.

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Thm (Friedman). An ω-model satisfies RCA_0 iff it is a Turing ideal.

$[X]^n$ is the set of unordered n-tuples of elements of X.

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RT^2_2 is more interesting.

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Cholak, Jockusch, and Slaman asked: Does SRT^2_2 imply RT^2_2?

Equivalently, does SRT^2_2 imply COH?

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RT^2_2 and SRT^2_2

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Open Question. Can this separation happen in ω-models?

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A $\widetilde{\Pi}^0_3$ formula is one of the form $\forall X\, \varphi(X)$, where φ is Π^0_3.

Thm (Patey and Yokoyama). $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is $\widetilde{\Pi}^0_3$-conservative over RCA_0.

Consider a principle

$$P \equiv \forall X\,[\Theta(X) \;\rightarrow\; \exists Y\,\Delta(X,Y)]$$

with Θ and Δ arithmetic.

We think of P as a problem.

An instance of this problem is an X such that $\Theta(X)$ holds.

A solution to this instance is a Y such that $\Delta(X,Y)$ holds.

P is computably reducible to Q, written as $P \leqslant_{\mathbf{c}} Q$, if

for every instance X of P,

there is an X-computable instance $\widehat{X}$ of Q s.t.,

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The uniform version is Weihrauch reducibility, $\leqslant_{\mathrm{W}}$.

Open Question. Is $\mathsf{COH} \leqslant_{\mathbf{c}} \mathsf{SRT}^2_2$?

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Thm (Dzhafarov). $\mathsf{COH} \nleqslant_{\mathbf{w}} \mathsf{SRT}^2_2$.

Open Question. Is COH $\leqslant_c \mathsf{SRT}^2_2$?

Thm (Dzhafarov). COH $\not\leqslant_w \mathsf{SRT}^2_2$.

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SRT^2_2 and COH

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Strong omniscient computable reducibility

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Open Question. Is $\mathsf{COH} \leqslant_{\mathrm{oc}} \mathsf{RT}^1_2$?

Thm (Patey; Hirschfeldt and Jockusch). $\mathsf{RT}^1_3 \nleqslant_{\mathrm{soc}} \mathsf{RT}^1_2$.

HT: For every finite coloring of $\mathbb{N}$, there is an infinite $S \subseteq \mathbb{N}$ s.t. every sum of distinct elements of S has the same color.

Hindman's Theorem

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Thm (Blass, Hirst, and Simpson). HT is provable in ACA_0^+ and implies ACA_0.

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Open Question (Hindman, Leader, and Strauss). Is there a proof of $\mathsf{HT}^{\leqslant 2}$ that is not a proof of HT?

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Thm (Blass, Hirst, and Simpson). HT is provable in ACA_0^+ and implies ACA_0.

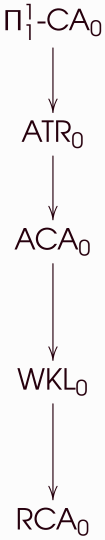
Open Question. Does $\mathrm{ACA}_0 \vdash \mathrm{HT}$?

$\mathrm{HT}^{\leqslant n}$ Is HT for sums of at most n elements.

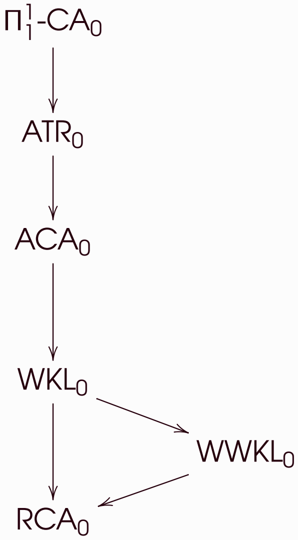
Open Question (Hindman, Leader, and Strauss). Is there a proof of $\mathrm{HT}^{\leqslant 2}$ that is not a proof of HT?

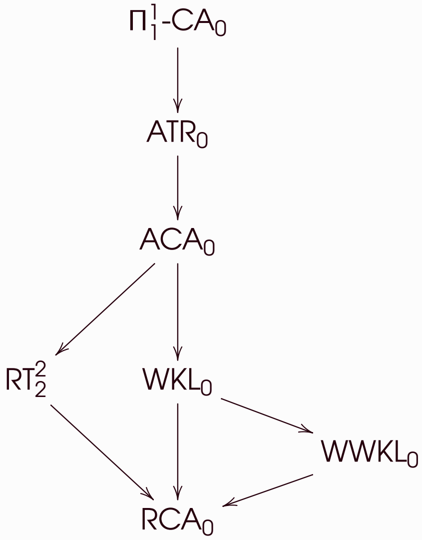
Open Question. Does $\mathrm{HT}^{\leqslant 2}$ imply HT, say over RCA_0?

A picture of the reverse mathematical universe

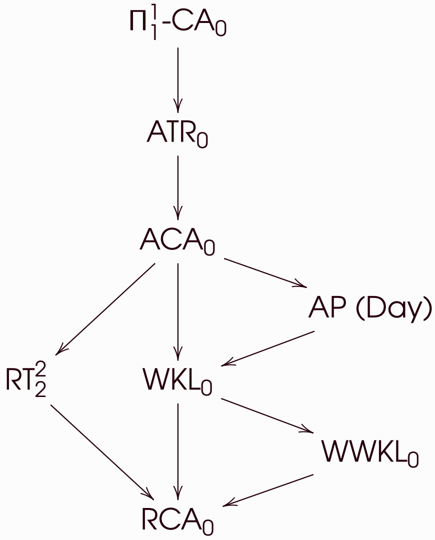


A picture of the reverse mathematical universe

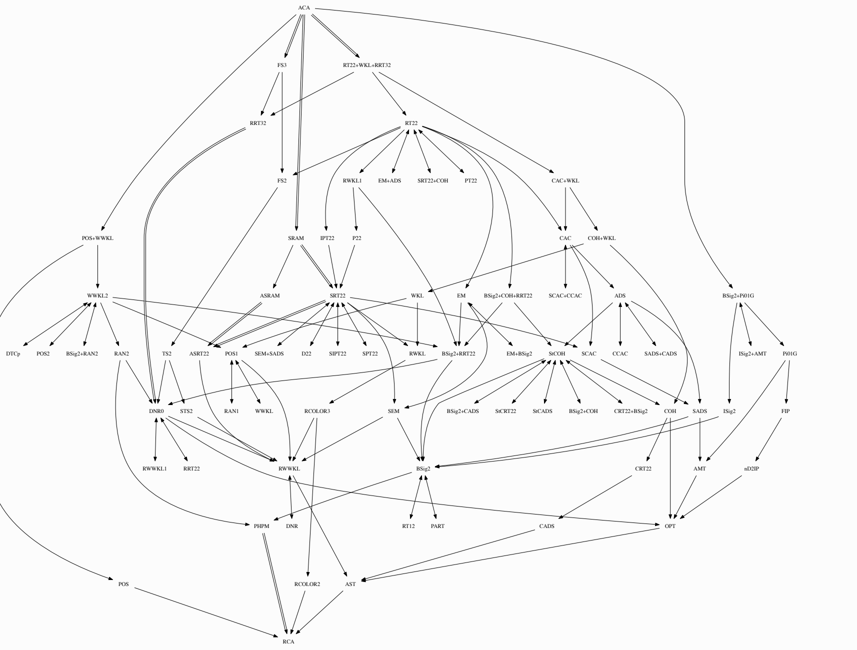



Π^1_1-CA_0
ATR_0
ACA_0
RT^2_2
WKL_0
$WWKL_0$
RCA_0

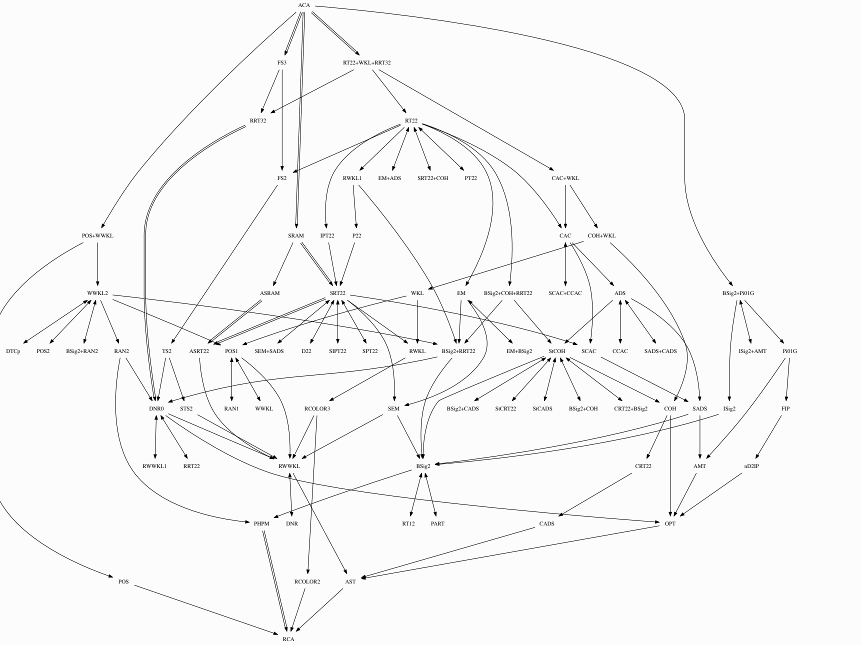
A picture of the reverse mathematical universe



Another picture of the reverse mathematical universe



Another picture of the reverse mathematical universe



Computability Theory: A Jaunt

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