Structure and enumeration theorems for hereditary properties in finite relational languages

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Logical 0-1 laws

Fix a finite first-order language $\mathcal{L}$. For each n suppose $K(n)$ is a set of $\mathcal{L}$-structures with universe $[n] = \{1, \ldots, n\}$. Set $K = \bigcup_{n \in \mathbb{N}} K(n)$.

Definition

We say K has a 0-1 *law* if for every $\mathcal{L}$-sentence ϕ,

$$\mu(\phi) = \lim_{n \to \infty} \frac{|\{G \in K(n) : G \models \phi\}|}{|K(n)|}$$

is either 0 or 1.

The *almost sure theory* of K is $T_{as}(K) = \{\phi \in \mathcal{L} : \mu(\phi) = 1\}$.

If K has a 0-1 law, then $T_{as}(K)$ is complete.

Discrete Metric Spaces

This is joint work with D. Mubayi.

Definition

Let $r \geq 2$ be an integer.

1. $M_r(n)$ is the set of metric spaces with underlying set $[n]$ and distances all in $[r]$.
2. $\mathcal{L}_r = \{d_1, \ldots, d_r\}$ where each d_i is a binary relation symbol.

Every $G \in M_r(n)$ is naturally an $\mathcal{L}_r$-structure:
for all $a, b \in G$, interpret

$$G \models d_i(a, b) \Leftrightarrow d(a, b) = i \text{ in } G.$$

Question

Does $M_r := \bigcup_{n \in \mathbb{N}} M_r(n)$ have a 0-1 law?

Answer for r even

Assume $r \geq 2$ is even. We now define a special subfamily of $M_r(n)$.

Definition

Let $C_r(n) = \{G \in M_r(n) : \text{for all } a \neq b \in [n], d(a,b) \in \{\frac{r}{2}, \ldots r\}\}$.

Theorem (Mubayi, T.)

$$\lim_{n \to \infty} \frac{|C_r(n)|}{|M_r(n)|} = 1.$$

Let $C_r = \bigcup_{n \in \mathbb{N}} C_r(n)$. A standard argument shows C_r has a 0-1 law.

Corollary (Mubayi, T.)

M_r has a 0-1 law and $T_{as}(M_r) = T_{as}(C_r)$.

Idea: Precise structure theorem + 0-1 law for $C_r \Rightarrow$ new 0-1 law for M_r.

How do we prove the precise structure theorem?

Key tool: approximate structure and enumeration.

Definition

Given $\delta > 0$ and two elements $G, G' \in M_r(n)$, we say G and G' are *δ-close* if $\left|\left\{ab \in \binom{[n]}{2} : d^G(a,b) \neq d^{G'}(a,b)\right\}\right| \leq \delta\binom{n}{2}$.

Let $C_r^\delta(n) = \{G \in M_r(n) : G \text{ is } \delta\text{-close to an element of } C_r(n)\}$.

Theorem (Mubayi, T.)

Structure: for all $\delta > 0$, there is $\beta > 0$ such that for large n,

$$\frac{|M_r(n) \setminus C_r^\delta(n)|}{|M_r(n)|} \leq 2^{-\beta\binom{n}{2}}.$$

Enumeration: $|M_r(n)| = |C_r(n)|(1 + 2^{o(n^2)}) = (\frac{r}{2}+1)^{\binom{n}{2}+o(n^2)}$.

Outline of 0-1 law

Approximate Structure and Enumeration $+$ Ad-hoc arguments

$$\Downarrow$$

Exact Structure and Enumeration ($C_r(n)$ takes over)

$$+$$

0-1 law for less complicated family ($C_r(n)$)

$$\Downarrow$$

0-1 law for complicated family ($M_r(n)$)

Examples where one can apply this strategy:

1. K_ℓ-free graphs (Kolaitis-Prömel-Rothschild)
2. T_k-free digraphs (Kühn-Osthus-Townsend-Zhao) + (Koponen)
3. Triangle-free 3-uniform hypergraphs (Balogh-Mubayi) + (Koponen)
4. Discrete metric spaces (Mubayi-T.)

Focus on Approximate Structure and Enumeration

Approximate structure and enumeration results ________.

- are important tools in proofs of certain 0-1 laws.
- are of independent interest in extremal combinatorics.
- have been proven for lots of combinatorial objects (e.g. graphs, digraphs, hypergraphs, colored hypergraphs, metric spaces).
- often have similar proofs using combination of:

1. extremal results
2. supersaturation results
3. stability theorems
4. graph removal lemmas
5. regularity lemmas
6. hypergraph containers theorem (Balogh-Morris-Samotij, Saxton-Thomason).

Question

Question

Is there a way to view these results (and their proofs) as examples of a general theorem (and its proof)?

Today: yes.

Main Ingredients:

- Hypergraph containers theorem (Balogh-Morris-Samotij, Saxton-Thomason).
- Triangle Removal for $\mathcal{L}$-structures (Aroskar-Cummings).
- Many combinatorics papers which have made the pattern of proof clear. Particularly recent work using the hypergraph containers theorem.

Remark: similar results were obtained independently by Falgas-Ravry, O'Connell, and Uzzell.

Hereditary $\mathcal{L}$-properties

Let $\mathcal{L}$ be a finite relational language and $\mathcal{H}$ a class of finite $\mathcal{L}$-structures. $\mathcal{H}$ has the *hereditary property* if $A \in \mathcal{H}$ and $B \subseteq_{\mathcal{L}} A$ implies $B \in \mathcal{H}$.

Definition

$\mathcal{H}$ is a *hereditary $\mathcal{L}$-property* if it has the hereditary property and is closed under isomorphism.

In the appropriate language, most of the results we want to generalize are for hereditary $\mathcal{L}$-properties.

Enumeration and structure of hereditary properties in the setting of graphs and other combinatorial structures have been studied in combinatorics.

Setup

For the rest of the talk,

- $\mathcal{L}$ is a fixed, finite relational language.
- r is the maximum arity of the relation symbols in $\mathcal{L}$.
- Assume $\mathcal{H}$ is a hereditary $\mathcal{L}$-property.
- For each n, $\mathcal{H}_n$ is the set of elements in $\mathcal{H}$ with domain $[n]$.
- For all n, $\mathcal{H}_n \neq \emptyset$.

Questions

1. $|\mathcal{H}_n| = ??$
2. What is the approximate structure of $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$?

$\mathcal{L}_{\mathcal{H}}$-structures

Definition

$S_r(\mathcal{H})$ is the set of complete, quantifier-free $\mathcal{L}$-types $p(x_1, \ldots, x_r)$ s.t. for each $i \neq j$, $x_i \neq x_j \in p(\bar{x})$ and $p(\bar{x})$ is realized in some element of $\mathcal{H}$.

Definition

$\mathcal{L}_{\mathcal{H}} = \{R_p(x_1, \ldots, x_r) : p(x_1, \ldots, x_r) \in S_r(\mathcal{H})\}$.

Notation: $V^{\underline{\ell}} = \{(a_1, \ldots, a_\ell) \in V^\ell : a_i \neq a_j \text{ each } i \neq j\}$.

Definition

Let V be a set. An $\mathcal{L}_{\mathcal{H}}$-structure M with domain V is an $\mathcal{L}_{\mathcal{H}}$-*template* if

- For all $\bar{a} \in V^{\underline{r}}$, there is $R_p(\bar{x}) \in \mathcal{L}_{\mathcal{H}}$ such that $M \models R_p(\bar{a})$.
- For all $p, q \in S_r(\mathcal{H})$, if $p(x_1, \ldots, x_r) = q(x_{\mu(1)}, \ldots, x_{\mu(r)})$ for some permutation μ of $[r]$, then for all $(a_1, \ldots, a_r) \in V^{\underline{r}}$,

$$M \models R_p(a_1, \ldots, a_r) \text{ if and only if } M \models R_q(a_{\mu(1)}, \ldots, a_{\mu(r)}).$$

Example

$\mathcal{L} = \{E(x, y)\}$ and $\mathcal{H}$ is the class of all finite triangle-free graphs.

$p(x, y) =$ the complete q.f. type containing $E(x, y) \wedge E(y, x) \wedge x \neq y$.

$q(x, y) =$ the complete q.f. type containing $\neg E(x, y) \wedge \neg E(y, x) \wedge x \neq y$.

Then $S_r(\mathcal{H}) = \{p(x, y), q(x, y)\}$ and $\mathcal{L}_{\mathcal{H}} = \{R_p(x, y), R_q(x, y)\}$.

Draw Example.

Subpatterns

Suppose M is an $\mathcal{L}_{\mathcal{H}}$-template with domain V.

Definition

An $\mathcal{L}$-structure N is a *full subpattern* of M if $dom(N) = V$ and for all $\{a_1, \ldots, a_r\} \in \binom{V}{r}$,

$$\text{if } p(x_1, \ldots, x_r) = qftp^N(a_1, \ldots, a_r) \text{ then } M \models R_p(a_1, \ldots, a_r).$$

In this case, write $N \trianglelefteq_p M$.

Observe any $\mathcal{L}$-structure N with domain V is determined by $qftp^N(a_1, \ldots, a_r)$ for all $\{a_1, \ldots, a_r\} \in \binom{V}{r}$.

Example.

Extremal Structures

Definition

An $\mathcal{L}_{\mathcal{H}}$-template M is called $\mathcal{H}$*-random* if $N \trianglelefteq_p M$ implies $N \in \mathcal{H}$.

- $\mathcal{R}([n], \mathcal{H})$ is the set of $\mathcal{H}$-random $\mathcal{L}_{\mathcal{H}}$-templates with domain $[n]$.
- $sub(M) = |\{N : N \trianglelefteq_p M\}|$.
- $\mathrm{ex}(n, \mathcal{H}) = \max\{sub(M) : M \in \mathcal{R}([n], \mathcal{H})\}$.

Definition

$M \in \mathcal{R}([n], \mathcal{H})$ is *extremal* if $sub(M) = \mathrm{ex}(n, \mathcal{H})$.

Observation: $|\mathcal{H}_n| \geq \mathrm{ex}(n, \mathcal{H})$.

- $\mathcal{R}_{\mathrm{ex}}([n], \mathcal{H})$ is the set of extremal M in $\mathcal{R}([n], \mathcal{H})$.

Results: Enumeration

Recall $\mathrm{ex}(n, \mathcal{H}) = \max\{sub(M) : M \in \mathcal{R}([n], \mathcal{H})\}$.

The *asymptotic density* of $\mathcal{H}$ is $\pi(\mathcal{H}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{H})^{1/\binom{n}{r}}$.

Theorem (T.)

For all hereditary $\mathcal{L}$-properties $\mathcal{H}$, $\pi(\mathcal{H})$ exists.

Theorem (T.)

If $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, then the following hold.

1. *If $\pi(\mathcal{H}) > 1$, then $|\mathcal{H}_n| = \pi(\mathcal{H})^{\binom{n}{r}+o(n^r)}$.*
2. *If $\pi(\mathcal{H}) \leq 1$, then $|\mathcal{H}_n| = 2^{o(n^r)}$.*

Distance Between First-Order Structures

Definition

Suppose $\mathcal{L}_0$ is a finite relational language with maximum arity r. If M and N are finite $\mathcal{L}_0$-structures with domain V. Set

$$\mathrm{dist}(M,N) = \frac{|\{A \in \binom{V}{r} : qftp^M(A) \neq qftp^N(A)\}|}{\binom{|V|}{r}}.$$

We say M and N are δ-*close* if $\mathrm{dist}(M,N) \leq \delta$.

This is basically the same as a definition of Aroskar-Cummings.

Stability and Structure

Definition

$\mathcal{H}$ has a *stability theorem* if for all $\delta > 0$ there is $\epsilon > 0$ such that for sufficiently large n the following holds. For all $M \in \mathcal{R}([n], \mathcal{H})$

$$sub(M) \geq \text{ex}(n, \mathcal{H})^{1-\epsilon} \Rightarrow M \text{ is } \delta\text{-close to an element of } \mathcal{R}_{ex}([n], \mathcal{H}).$$

Let $E(n, \mathcal{H}) = \{G \in \mathcal{H}_n : G \trianglelefteq_p M, \text{ some } M \in \mathcal{R}_{\text{ex}}([n], \mathcal{H})\}$.
Let $E^\delta(n, \mathcal{H}) = \{G \in \mathcal{H}_n : G \text{ is } \delta\text{-close to some } G' \in E(n, \mathcal{H})\}$.

Theorem (T.)

Suppose $\pi(\mathcal{H}) > 1$ and $\mathcal{H}$ has a stability theorem. Then for all $\delta > 0$, there is a $\beta > 0$ such that for sufficiently large n,

$$\frac{|\mathcal{H}_n \setminus E^\delta(n, \mathcal{H})|}{|\mathcal{H}_n|} \leq 2^{-\beta\binom{n}{r}}.$$

Back to Metric Spaces

Given $r \geq 2$, let $\mathcal{L} = \{d_1, \ldots, d_r\}$, and let $\mathcal{M}_r$ be the hereditary $\mathcal{L}$-property obtained by closing $\bigcup_{n \in \mathbb{N}} M_r(n)$ under isomorphism.

Fact

If r is even then the following hold.

- $E(n, \mathcal{M}_r) = C_r(n)$, so for all $\delta > 0$, $E^\delta(n, \mathcal{M}_r) = C_r^\delta(n)$.
- $\pi(\mathcal{M}_r) = (\frac{r}{2} + 1)$.

Consequently, our general counting theorem implies the following.

Theorem

When $r \geq 2$ is even, $|M_r(n)| = \pi(\mathcal{M}_r)^{\binom{n}{2} + o(n^2)} = (\frac{r}{2} + 1)^{\binom{n}{2} + o(n^2)}$.

Back to Metric Spaces

Fact

If $r \geq 2$ is even then $\mathcal{M}_r$ has a stability theorem.

Consequently, our general structure theorem implies the following.

Theorem

Let $r \geq 2$ be even. For all $\delta > 0$, there is $\beta > 0$ such that for large n,

$$\frac{|(\mathcal{M}_r)_n \setminus E^\delta(n, \mathcal{M}_r)|}{|(\mathcal{M}_r)_n|} = \frac{|M_r(n) \setminus C_r^\delta(n)|}{|M_r(n)|} \leq 2^{-\beta\binom{n}{2}}.$$

Back to Metric Spaces

Recall that when r is even, $\bigcup_{n\in\mathbb{N}} M_r(n)$ has a 0-1 law (Mubayi-T.).

Conjecture (Mubayi-T.)

When r is odd, $\bigcup_{n\in\mathbb{N}} M_r(n)$ does not have a 0-1 law.

Question: What is the difference between the even and odd cases?

Theorem (Mubayi-T., T.)

When $r \geq 2$ is even, $\mathcal{M}_r$ has a stability theorem. When r is odd, $\mathcal{M}_r$ does not have a stability theorem.

A Conjecture

This naturally leads to the following conjecture.

Assume $\mathcal{L}$ is a finite relational language.

Definition

Suppose $\mathcal{H}$ is a collection of finite $\mathcal{L}$-structures such that $\mathcal{H}_n \neq \emptyset$ for each *n*. Then $\mathcal{H}$ *has a* 0-1 *law* if $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ has a 0-1 law.

Conjecture (T.)

If $\mathcal{H}$ is a hereditary $\mathcal{L}$-property with $\pi(\mathcal{H}) > 1$ and a 0-1 law, then $\mathcal{H}$ has a stability theorem.

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