Free Logic: its formalization and some applications

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What is "Free Logic"?

*"Free logic" is logic free of **existential presuppositions** in general and with respect to **singular terms** in particular.*

To be discussed:

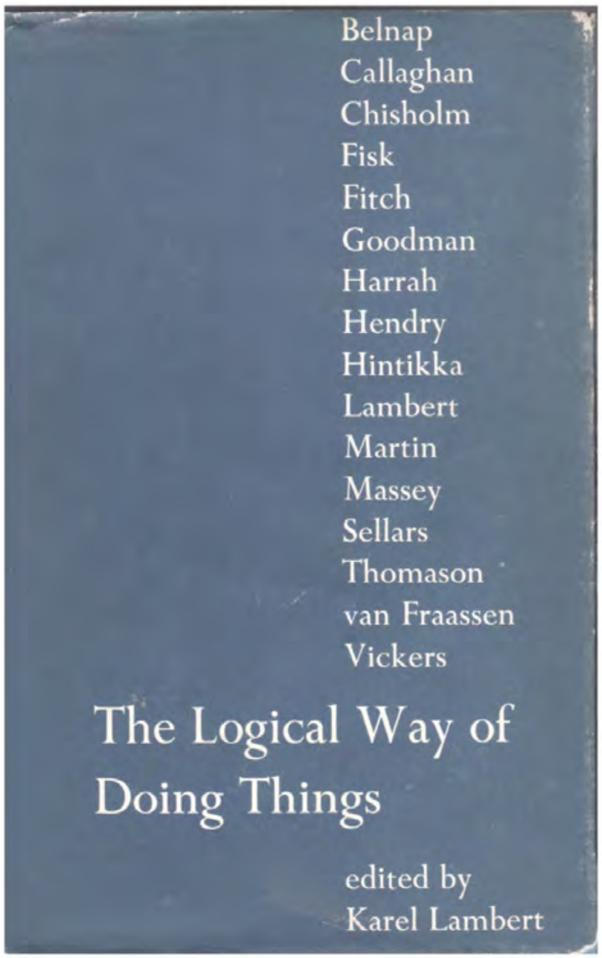
Formalization
Empty domains
Descriptions
Virtual classes
Structures with partial functions, and
Models (classical and intuitionistic)

Prof. J. Karel Lambert (Emeritus, UC Irvine) gave the subject its name and its profile as a well defined field of research some 60 years ago:

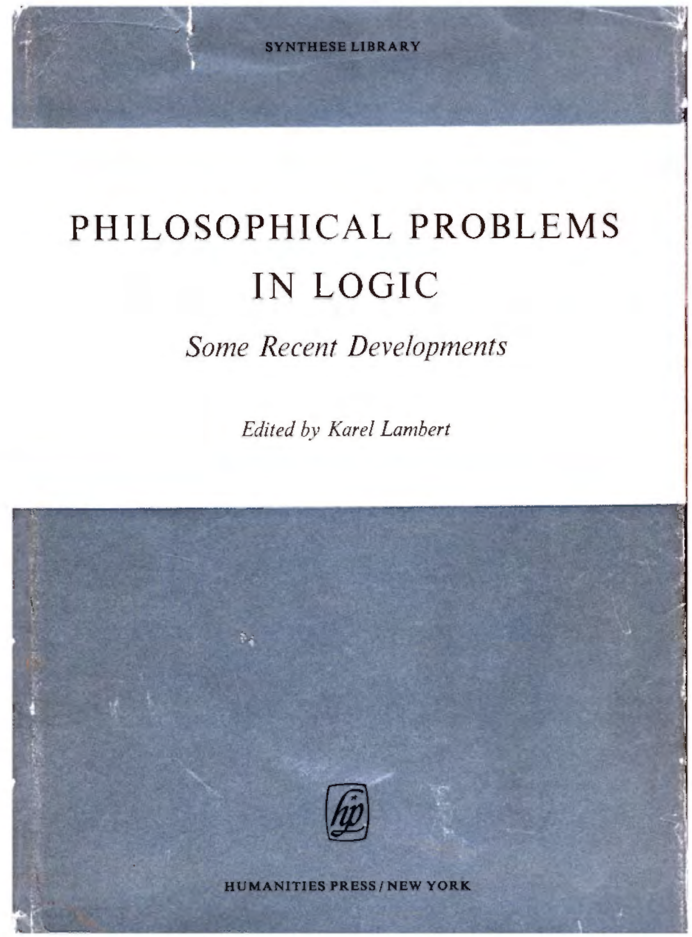
Karel Lambert. "Notes on E!." ***Philosophical Studies***, vol. 9 (1958), pp. 60-63.

Karel Lambert. "Singular Terms and Truth." ***Philosophical Studies***, vol. 10 (1959), pp. 1-5.

Lambert's Collections (I)

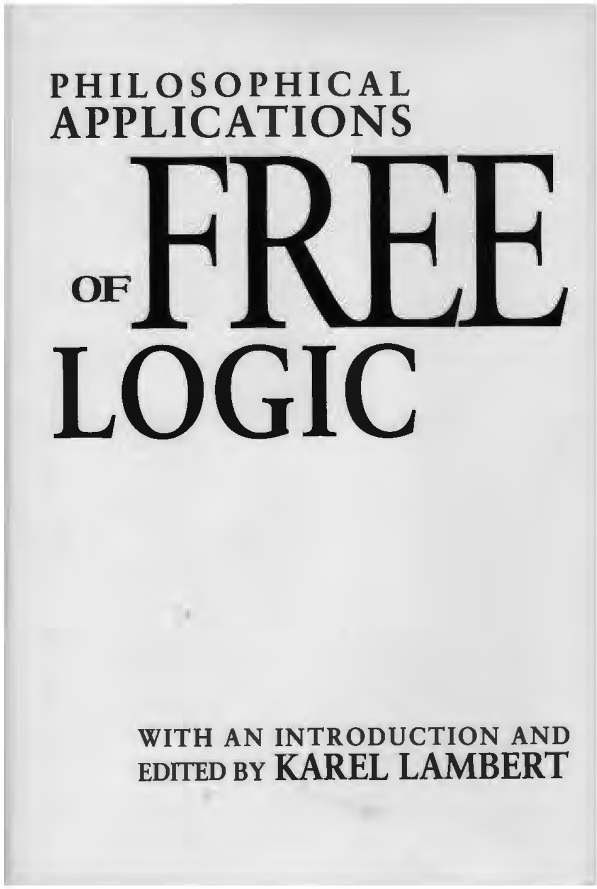


Karel Lambert (ed.) **The Logical Way of Doing Things**, Yale University Press, 1969, xiii + 325 pp.

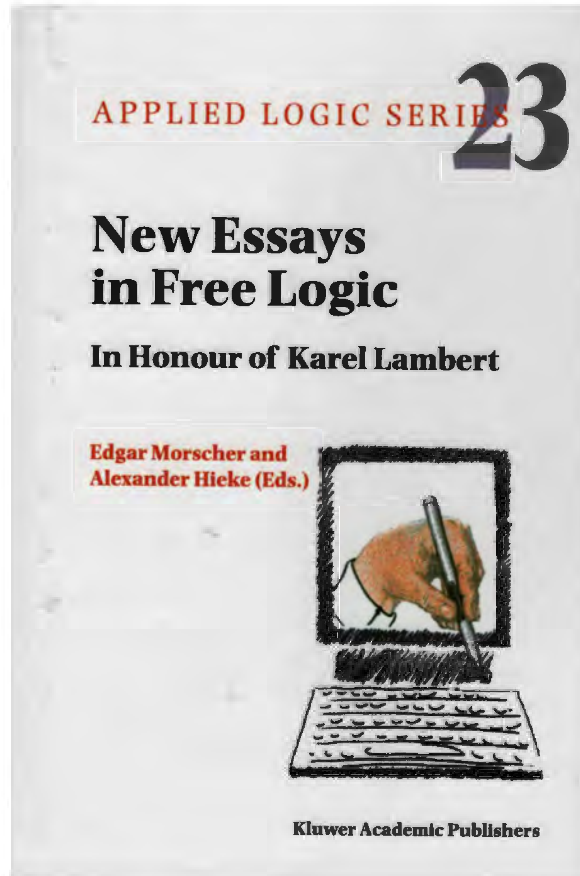


Karel Lambert (ed.) **Philosophical Problems in Logic: Some Recent Developments**, D. Reidel Publishing Co., 1970, vi + 176 pp.

Lambert's Collections (II)

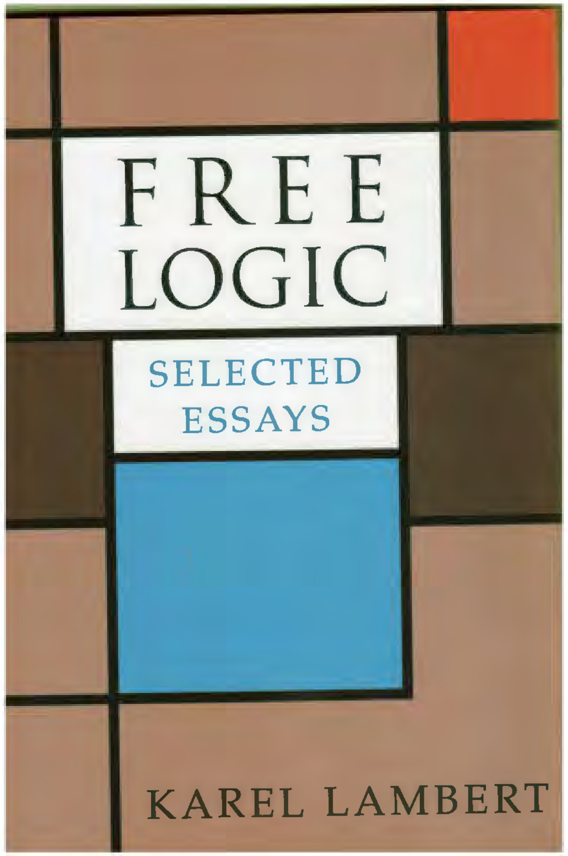


Karel Lambert (ed.) **Philosophical Applications of Free Logic**, Oxford University Press, 1991, x + 309 pp.



Edgar Morscher, Alexander Hieke (eds.) **New Essays in Free Logic: In Honor of Karel Lambert**, Kluwer Academic Publishers, 2001, vii + 255 pp.

Lambert's Collections (III)



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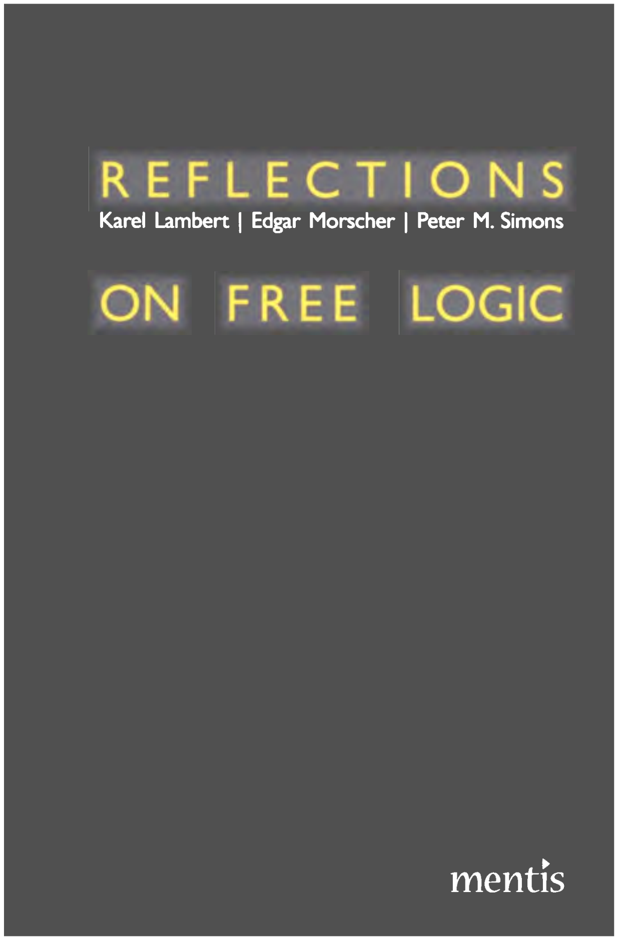
5. Foundations of the Hierarchy of Positive Free Definite Description Theories
6. Predication and Extensionality
7. Nonextensionality
8. The Philosophical Foundations of Free Logic
9. Logical Truth and Microphysics

Karel Lambert. **Free Logic: Selected Essays**,
Cambridge University Press, 2003, xii + 191 pp.

Lambert's Collections (IV)



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Karel Lambert. *Speaking Freely*

Karel Lambert. *Extensionality, Bivalence and Singular Terms like "The Greatest Natural Number"*

Karel Lambert. *The*

Karel Lambert. *Dialogue with Edgar Morscher*

Peter M. Simons. *Higher-Order Logic–Free of Ontological Commitments*

Edgar Morscher. *The Trouble with Existentials, in particular with Singular Existentials*

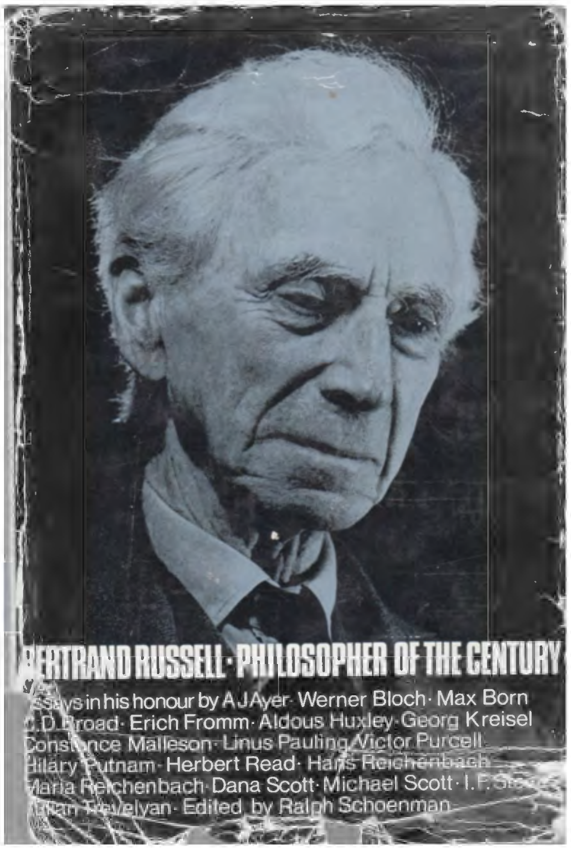
Edgar Morscher. *Should the Quantifiers in Free Logic be allowed to lose their Existential Import?*

A Systematic and Historical Survey of Free Logic

Annotated Bibliography of Works of Free Logic

Karel Lambert, Edgar Morscher and Peter M. Simons.
Reflections on Free Logic,
Mentis Verlag, Münster, 2017, 149 pp.

Scott's Presumptuousness



Contributions by:

A.J. Ayer	Werner Bloch
Max Born	C. D. Broad
Erich Fromm	Aldous Huxley
Georg Kreisel	Constance Malleson
Linus Pauling	Victor Purcell, CMG
Hilary Putnam	W. V. Quine
Herbert Read	Hans Reichenbach
Maria Reichenbach	Dana Scott
Rev. Michael Scott	I. F. Stone

Julian Trevelyan

Ralph Schoenman (ed.) **Bertrand Russell, Philosopher of the Century: Essays in his Honour**, George Allen & Unwin LTD, 1967, 326 pp.

Selected Scott Writings

Dana Scott. "Existence and description in formal logic." In: **Bertrand Russell: Philosopher of the Century**, Ralph Schoenman (ed.), George Allen & Unwin, London, 1967, pp. 181–200. Reprinted with additions in: **Philosophical Applications of Free Logic**, K. Lambert (ed.), Oxford Universitry Press, 1991, pp. 28–48.

Alonzo Church. Review of the above. **The Journal of Symbolic Logic**, vol. 38 (1973), pp. 166-169.

Dana Scott. "Extending the topological interpretation to intuitionistic analysis." Part I: **Composito Mathematica**, vol. 20 (1968), 194-210. Part II. In: **Intuitionism and Proof Theory** (eds. Myhill et al.), North-Holland, 1970, pp. 235-255.

Dana Scott. "Identity and existence in intuitionistic logic." In: **Applications of Sheaves, Durham Proceedings 1977**, M. Fourman and D. Scott (eds.), Springer-Verlag, LNM 753 (1979), pp. 660–696.

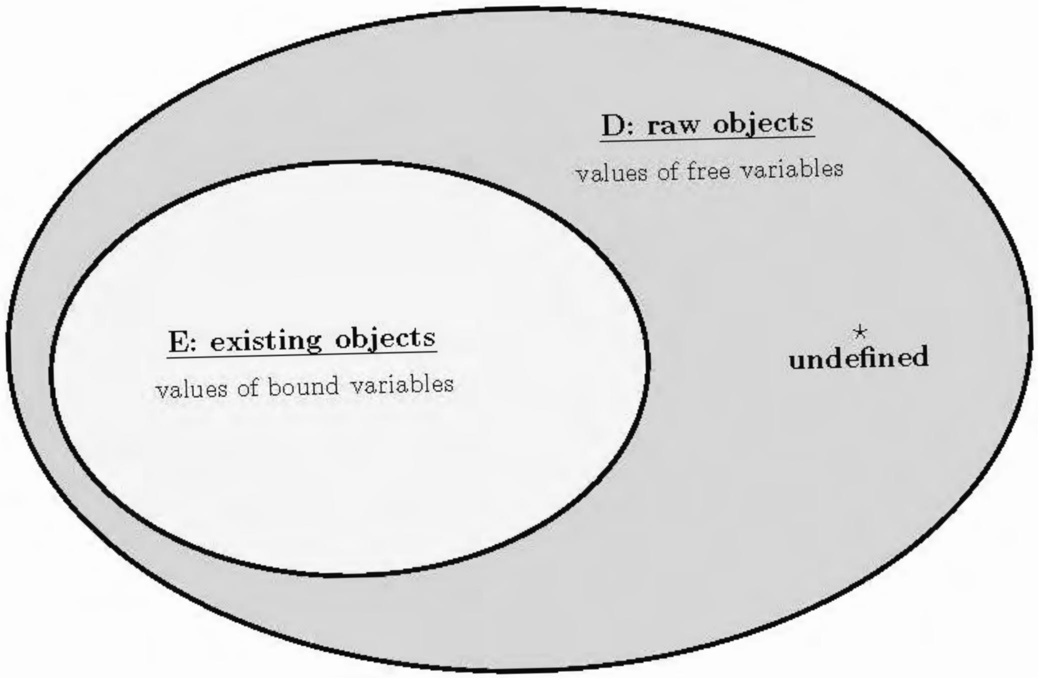
Michael Fourman and Dana Scott. "Sheaves and logic." In: **Applications of Sheaves, Durham Proceedings 1977**, Fourman and Scott (eds.), Springer-Verlag, LNM, vol. 753 (1979), pp. 302–401.

Dana Scott. "The Algebraic Interpretation of Quantifiers: Intuitionistic and Classical." In: **Seventy Years of Foundational Studies: A Tribute to Andrzej Mostowski**, A. Ehrenfeucht et al. (eds.), IOS Press, 2008. pp. 289–312.

These papers, together with the talk slides and other background materials, have been put online via:

`http://logic.berkeley.edu/past-events.html`

Inner and Outer Domain Conventions



Bound variables range over **E** (which may be ***empty***).

Free variables range over **D** containing **E** (and is ***non empty***).

If desired, a ***special object*** ★ can be designated in **D** \ **E**.

The ***equivalence relation*** ≡ is the ***identity*** on E but ***collapses*** **D** \ **E**.

Rules for Variables and Quantifiers

(Sub) $$\frac{\Phi(x)}{\Phi(\tau)}$$

(UI) $$\forall x.\Phi(x) \wedge \mathbf{E}x \Longrightarrow \Phi(x)$$

(UG) $$\frac{\Phi \wedge \mathbf{E}x \Longrightarrow \Psi(x)}{\Phi \Longrightarrow \forall x.\Psi(x)}$$

(EI) $$\Phi(x) \wedge \mathbf{E}x \Longrightarrow \exists x.\Phi(x)$$

(EG) $$\frac{\Phi(x) \wedge \mathbf{E}x \Longrightarrow \Psi}{\exists x.\Phi(x) \Longrightarrow \Psi}$$

In the above x is a free **variable** and τ is any well formed **term**.
Substitution always has to be done under the **condition** that **no** free variables are **captured**.

Axioms for Equality with Collapsing

(Rep) $\Phi(x) \wedge x \equiv y \Longrightarrow \Phi(y)$

(Comp) $\forall z\, [\, x \equiv z \Longleftrightarrow y \equiv z\,] \Longrightarrow x \equiv y$

Easily Proved Conclusions

(Refl) $x \equiv x$

(Sym) $x \equiv y \Longrightarrow y \equiv x$

(Trans) $x \equiv y \wedge y \equiv z \Longrightarrow x \equiv z$

(Exist) $\mathbf{E}x \Longleftrightarrow \exists y\, [\, x \equiv y\,]$

(Id) $x = y \Longleftrightarrow \mathbf{E}x \wedge \mathbf{E}x \wedge x \equiv y$ (by definition)

(Equ) $x \equiv y \Longleftrightarrow [\, [\, \mathbf{E}x \vee \mathbf{E}y\,] \Longrightarrow x = y\,]$

If the collapse of D \ E is not desired, then just use the ordinary identity rules for = over the whole of D without (Comp).

Axioms for Definite Descriptions

(Desc) $y \equiv \mathbf{I}.\Phi(x) \Longleftrightarrow \forall x\, [\, x \equiv y \Longleftrightarrow \Phi(x)\,]$

Axioms **without** Collapse

(Desc) $\forall y\, [\, y = \mathbf{I}.\Phi(x) \Longleftrightarrow \forall x\, [\, x = y \Longleftrightarrow \Phi(x)\,]]$

(Star) $\neg \exists y \forall x\, [\, x = y \Longleftrightarrow \Phi(x)\,] \Longrightarrow \mathbf{I}.\Phi(x) = \bigstar$

Axioms for Strict Predicates and Total Functions

(StrP) $P(x,y) \Longrightarrow [\, \mathbf{E}x \wedge \mathbf{E}y\,]$

(StrF) $\mathbf{E}\, f(x,y) \Longrightarrow [\, \mathbf{E}x \wedge \mathbf{E}y\,]$

(TotF) $[\, \mathbf{E}x \wedge \mathbf{E}y\,] \Longrightarrow \mathbf{E}\, f(x,y)$

Note that when predicates or functions have **several** parameters, then **strictness conditions** can be **mixed** and apply only to some.

Axioms for Virtual Classes

Convention: **No** collapse of **D \ E** allowed **!!**.

(Ext) $\forall z\,[\,z \in x \iff z \in y\,] \Longrightarrow x = y$

This means **everything** is a class.

(Abs) $y \in \{\,x \mid \Phi(x)\} \iff \mathbf{E}y \wedge \Phi(y)$

This implies **abstraction** binds only to **existing** things.

Easily Proved Conclusions

(Self) $\{\,x \mid x \in y\,\} = y$

(Mem) $x \in y \Longrightarrow \mathbf{E}x$

(Equ) $\forall x\,[\,\Phi(x) \iff \Psi(x)\,] \Longrightarrow \{\,x \mid \Phi(x)\} = \{\,x \mid \Psi(x)\}$

(Rus) $\neg\,\mathbf{E}\,\{\,x \mid \neg\, x \in x\}$

If **atoms** are desired, assume they are of the form $\{a\} = a$.

Puzzle: Can such atoms be **non-existents**?

Axioms for Partial Combinatory Algebras

(Appl)	$\mathbf{E}\, x(y) \Longrightarrow [\, \mathbf{E}x \wedge \mathbf{E}y \,]$
(K)	$\mathbf{E}\, \mathbf{K} \wedge \forall x.\, \mathbf{E}\, \mathbf{K}(x)$
(S)	$\mathbf{E}\, \mathbf{S} \wedge \forall x, y.\, \mathbf{E}\, \mathbf{S}(x)(y)$
(K≡)	$\mathbf{E}\, y \Longrightarrow \mathbf{K}(x)(y) \equiv x$
(S≡)	$\mathbf{S}(x)(y)(z) \equiv x(z)(y(z))$

Example: $x(y)$ is defined as $\{x\}(y)$. (Kleene's PCA)

Theorem. Not every partial combinatory algebra can be extended to a total combinatory algebra, but the Kleene PCA is extendable — with the right choice of $\mathbf{S}$ and $\mathbf{K}$.

Inge Bethke, Jan Willem Klop, and Roel De Vrijer.

"Completing partial combinatory algebras with unique head-normal forms."
IEEE LICS Proceedings (1996), pp. 448–454.

"Extending partial combinatory algebras."
Mathematical Structures in Compter Science, vol. 9 (1999), pp. 483–505.

Ω-sets and Intuitionistic Logic

Definition. A complete lattice Ω is a ***complete Heyting algebra*** (cHa) iff it has an ***implication*** operation →, where always $(p \wedge q) \leq r$ iff $p \leq (q \to r)$.

Note: This condition is equivalent to distributing ***finite*** meets over ***all*** joins.

Example: The ***open subsets*** of any topological space always form a cHa.

Definition. An Ω-***set*** is a structure **A** together with an Ω-valued ***equality***, $[\![\bullet = \bullet]\!]$, where, for all a, b, and c in A we have:

$$[\![a = b]\!] = [\![b = a]\!] \text{ and } ([\![a = b]\!] \wedge [\![b = c]\!]) \leq [\![a = c]\!].$$

Additionally we define:

$$[\![\mathbf{E}a]\!] = [\![a = a]\!] \text{ and } [\![a \equiv b]\!] = ([\![\mathbf{E}x]\!] \vee [\![\mathbf{E}y]\!]) \to [\![a = b]\!].$$

Theorem. Extending the $[\![\bullet]\!]$-notation to all logical formulae over any Ω-set validates all the laws of Free Intuitionistic Logic.

Complete Ω-sets and Total Elements

Definition. A ***total element*** of an Ω-set A is an a in A where $[\![a = a]\!] = \top$

Definition. A ***singleton*** for an Ω-set A is a mapping $s: A \longrightarrow \Omega$, where for all a, b in A we have

$$(s(a) \wedge [\![a = b]\!]) \leq s(b) \text{ and } (s(a) \wedge s(b)) \leq [\![a = b]\!].$$

Example: Given c in A, then the function $s(a) = [\![a = c]\!]$ is a singleton.

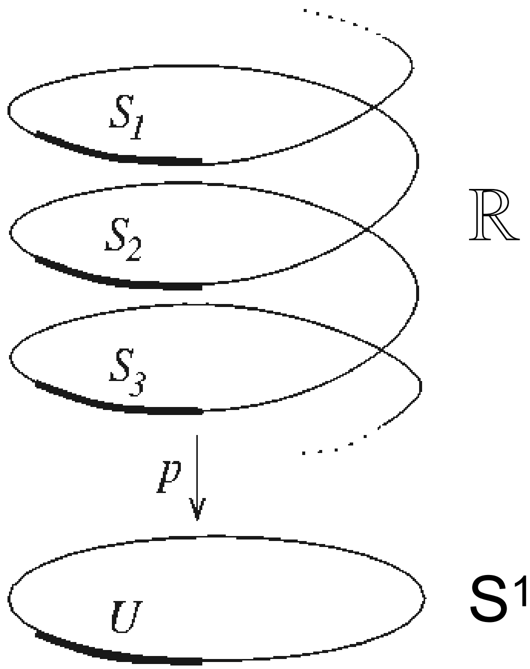
Example: The ***constant function*** $s(a) = \bot$ is also a singleton.

Definition. A ***complete Ω-set*** is a structure A where for every singleton $s: A \longrightarrow \Omega$ there is a ***unique*** a in A where, for all b in A, $s(b) = [\![a = b]\!]$.

Example: The ***constant function*** $\bot$ gives us one ***totally non-existent*** element ★.

Note: Complete Ω-sets may have many elements with a partial degree of existence but also have no total elements.

The Covering-Space Example



$p : \mathbb{R} \longrightarrow S^1$ is a ***local homeomorphism.***

$\Omega = \text{Open}(S^1)$

$A = \{\, a\colon U \longrightarrow \mathbb{R} \mid U \in \Omega \;\&\; p \circ a = \text{Id}_U \,\}$

$[\![\, a < b \,]\!] = \{\, t \in \mathbf{dom}(a) \cap \mathbf{dom}(b) \mid a(t) < b(t) \,\}$

$[\![\, \mathbf{E}a \,]\!] = U = \mathbf{dom}(a)$ is ***never*** be the whole of S^1.

For further examples see the ***Fourman-Scott paper*** and also:

Saunders Mac Lane and Ieke Moerdijk. **"Sheaves in Geometry and Logic: A First Introduction to Topos Theory."** Springer Verlag, 1992, xii + 627 pp.

John L. Bell. **"The Continuous and the Infinitesimal in Mathematics and Philosophy."** Polimetrica, Milano, 2005, 352 pp.

Mac Lane's First Category Theory Axioms

ML0. $\mathbf{E}(x \circ y) \Longrightarrow [\, \mathbf{E}\, x \wedge \mathbf{E}\, y \,]$

ML1. $\mathbf{E}((z \circ y) \circ x) \Longrightarrow \mathbf{E}(y \circ x)$

ML2. $\mathbf{E}(z \circ (y \circ x)) \Longrightarrow \mathbf{E}(z \circ y)$

ML3. $[\, \mathbf{E}(z \circ y) \wedge \mathbf{E}(y \circ x) \,] \Longrightarrow ((z \circ y) \circ x) = (z \circ (y \circ x))$

ML4. $\forall x\, \exists d\, [\, \mathbf{ID}(d) \wedge \mathbf{E}(x \circ d) \,]$

ML5. $\forall x\, \exists c\, [\, \mathbf{ID}(c) \wedge \mathbf{E}(c \circ x) \,]$

Definition. $\mathbf{ID}(y) \Longleftrightarrow \forall x\, [\, [\, \mathbf{E}(x \circ y) \Longrightarrow x \circ y = x \,] \wedge [\, \mathbf{E}(y \circ x) \Longrightarrow y \circ x = x \,] \,]$

Remark: Compare these axioms to a ***monoid*** with a ***unit element***.

Saunders Mac Lane. "Groups, categories and duality." **Proceedings of the National Academy of Sciences**, vol. 34 (1948), pp. 263–267.

Remark: ***Later definitions*** of a category were much more ***relaxed.***

Freyd and Scedrov's Category Theory Axioms

FS0.	$\mathbf{E}(x \circ y) \Longleftrightarrow \mathbf{dom}\ x = \mathbf{cod}\ y$
FS1.	$\mathbf{cod}(\mathbf{dom}\ x) \equiv \mathbf{dom}\ x$
FS2.	$\mathbf{dom}(\mathbf{cod}\ y) \equiv \mathbf{cod}\ y$
FS3.	$x \circ (\mathbf{dom}\ x) \equiv x$
FS4.	$(\mathbf{cod}\ y) \circ y \equiv y$
FS5.	$\mathbf{dom}(x \circ y) \equiv \mathbf{dom}((\mathbf{dom}\ x) \circ y)$
FS6.	$\mathbf{cod}(x \circ y) \equiv \mathbf{cod}(x \circ (\mathbf{cod}\ y))$
FS7.	$x \circ (y \circ z) \equiv (x \circ y) \circ z$

Peter J. Freyd and Andre Ščedrov. "**Categories, Allegories.**"
North-Hollnd Elsevier, 1990, xvii + 296 pp.

Remark: In the book, axiom **FS0** was ***misstated*** with ≡ instead of =

Scott's Category Theory Axioms

S1. $\mathbf{E}(\mathbf{dom}\ x) \Longrightarrow \mathbf{E}\ x$

S2. $\mathbf{E}(\mathbf{cod}\ y) \Longrightarrow \mathbf{E}\ y$

S3. $\mathbf{E}(x \circ y) \Longleftrightarrow \mathbf{dom}\ x = \mathbf{cod}\ y$

S4. $x \circ (\mathbf{dom}\ x) \equiv x$

S5. $(\mathbf{cod}\ y) \circ y \equiv y$

S6. $x \circ (y \circ z) \equiv (x \circ y) \circ z$

Remark: ***Axioms*** are as presented in the ***Fourman-Scott paper***.
But see the recent papers by ***Scott*** and ***Benzmüller*** about ***proofs***.

“Automating Free Logic in Isabelle/HOL. In: **Mathematical Software (ICMS 2016)**, Springer, LNCS, vol. 9725 (2016), pp. 43-50.

“Axiom Systems for Category Theory in Free Logic.” **Archive of Formal Proofs,** 2018.

“Automating Free Logic in HOL, with an Experimental Application in Category Theory.” **Journal of Automated Reasoning**, 2019. (online & in press), 11 pp.