The Class CAT of Locally Small Categories as a Functor-Free Framework for Foundations and Philosophy

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1. Outline

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2. Motivations for foundations and philosophy

- Q? Foundations: What rests on what? A: Nonwellfounded cycles ok
- Q? What is ZFC? Mathematics, metamathematics, term rewriting?
- Q? Can algebraic structures be constructed more economically?
  A: $\mathcal{CAT}$ to the rescue.
- Q? Is point-set topology fundamentally different from algebra?
  A: Each mirrors the other. (Descartes would have liked that.)
- Q? Of what possible use are categories without functors?
  A: Treating $\mathcal{CAT}$ as merely a class will get us a long way.
- Q? Are properties intrinsically intensional?
  A: We propose an extensional notion of "property".
- Q? Is "red" more a noun or an adjective?
  A: (C.I.Lewis) "Red" as a quale is equally noun and adjective.
- Q? How long must evolution of human consciousness take?
  A: $\mathcal{CAT}$ could accelerate natural selection.
- Q? Is the distinction between sort and property a fundamental feature of human consciousness? A: Open question.

3. The class CAT as an alternative to the theory ZFC

ZFC starts from the binary relation $\in$ of **set membership**.
$\mathcal{C}AT$ starts from the "binary" operation $\circ$ of **function composition**.

Sizes *Sets* : $\emptyset, \{\emptyset\}, \omega, \mathbb{Q}, c, \aleph_{\epsilon_{47}}$. *Classes*: **Set**, **Grp**. *"Even larger"*: $\mathcal{C}AT$.

$\mathcal{C}AT$ consists of all locally small categories. Each category $\mathcal{A} \in \mathcal{C}AT$ consists of a class of **objects** $X, Y, \ldots$ along with a set $\mathcal{A}(X, Y)$ (**homset**) of **morphisms** $f : X \to Y$ for each pair X, Y. Paths $X \xrightarrow{f} Y \xrightarrow{g} Z$ **compose** associatively as $gf : X \to Z$, with a morphism $1_X : X \to X$ at each object X as a two-sided **identity** for composition.

Examples. **Set**; **Grp**; monoids ($|ob(\mathcal{A})| = 1$); posets ($|\mathcal{A}(X, Y)| \leq 1$).

Isomorphism $f : X \to Y$: $\exists g : Y \to X$ s.t. $gf = 1_X$, $fg = 1_Y$.

Subcategory $\mathcal{A} \subseteq \mathcal{B}$, **Isomorphic** cats $\mathcal{A} \sim \mathcal{B}$: by analogy with groups.
Equivalence: $\mathcal{A} \equiv \mathcal{B}$ when $\mathcal{A}, \mathcal{B}$ have isomorphic subcategories $\mathcal{A}' \sim \mathcal{B}'$, every object $X \in \mathcal{A}$ is isomorphic to an object $X' \in \mathcal{A}'$, & likewise for $\mathcal{B}$.

4. The category **Set**: uniqueness in CAT (up to $\equiv$)

A **set-like category** $(\mathcal{A}, \mathbf{1})$ is a category $\mathcal{A} \in \mathcal{C}AT$ with an object $\mathbf{1}$ s.t.
(i) ($\mathbf{1}$ is the representative singleton) $|\mathcal{A}(\mathbf{1}, \mathbf{1})| = 1$, and
(ii) (extensionality of functions) for any two morphisms $f, g : X \to Y$ in $\mathcal{A}$, if for all morphisms $x : \mathbf{1} \to X$ (elements of X) $fx = gx$, then $f = g$.

The **carrier** of object X is the homset $\mathcal{A}(\mathbf{1}, X)$.

A set-like category $(\mathcal{A}, \mathbf{1})$ is **full** when it is its only set-like carrier-preserving extension (**CPE**) by morphisms alone. A full set-like category $(\mathcal{A}, \mathbf{1})$ is **complete** when it is equivalent to its every full set-like CPE.

Theorem

There exists a complete full set-like category. Call "it" **Set**.

Proof.

Take all the homsets in $\mathcal{C}AT$. □

5. Graphs: Irreflexive and reflexive

A **graph-like category** $(\mathcal{A}, V, E)$ is a category $\mathcal{A} \in \mathcal{C}AT$ with representative vertex V, and representative edge E having $s, t : V \to E$ as its two vertices. Each object G has vertices and edges drawn from carriers $\mathcal{A}(V, G)$ and $\mathcal{A}(E, G)$. All morphisms are graph homomorphisms, by associativity of composition (explained on next slide).

Define **full** and **complete** by analogy with set-like categories.

Theorem

There exists a complete full graph-like category. Call "it" **Grph**.

Proof.

For each pair of homsets in $\mathcal{C}AT$ form objects G s.t. $(\mathcal{A}(V, G), \mathcal{A}(E, G))$ is that pair, for all composites $\mathcal{A}(E, G)s$, $\mathcal{A}(E, G)t$ (incidences). □

For reflexive graphs **RGrph**, introduce $i : E \to V$ as vertex V's self-loop, thereby creating two self-loops $si, ti : E \to E$ one at each end of edge E.

6. Presheaves

Any category $\mathcal{A}$ and set $J \subseteq ob(\mathcal{A})$ of objects thereof determines the full subcategory $\mathcal{J} \subseteq \mathcal{A}$ such that $ob(\mathcal{J}) = J$. Call such a category a $\mathcal{J}$-like category. J indexes the carriers $\mathcal{A}(j, A)$ of each object A while the morphisms $f : k \to j$ of $\mathcal{J}$ index the unary operations $af : \mathcal{A}(j, A) \to \mathcal{A}(k, A)$ defined by the *right* action f_A of f on the elements a of $\mathcal{A}(j, A)$.

This makes each object A of such a pairing $(\mathcal{A}, J)$ a heterogeneous algebra, and each morphism $h : A \to B$ a homomorphism of algebras wrt those operations by its *left* action on $a \in \mathcal{A}(j, A)$, using associativity in $k \xrightarrow{f} j \xrightarrow{a} A \xrightarrow{h} B$ to show $h(f_A(a)) = f_B(h(a))$.

Presheaves are simply generalized graphs with $\{V, E\}$ generalized to J and operation symbols as the morphisms of $\mathcal{T} = \mathcal{J}^{op}$. As heterogeneous unary algebras they are models of the commutative diagrams (equations between terms) in $\mathcal{T}$. Set theory $= \bullet$, graph theory $= V \overset{s,t}{\Longleftarrow} E$, etc.

7. Chu(Set, K) as a universal framework

A **Chu space** (A, r, X) over a set K consists of sets A and X and a function $r : A \times X \to K$, i.e. matrix. A **Chu transform** $(h, h') : (A, r, X) \to (B, s, Y)$ is a pair $h : A \to B$, $h' : Y \to X$ of functions satisfying $s(h(a), y) = r(a, h'(y))$ for all $a \in A, y \in Y$.

As for **Set** etc., let $J = \{\mathbf{1}\}$ in $\mathcal{A}$, let $L = \{\perp\}$ be a second rigid object in $\mathcal{A}$, and let $\mathcal{A}(\mathbf{1}, \perp) = K$. For any object ArX of $\mathcal{A}$, take $A = \mathcal{A}(\mathbf{1}, ArX)$ to be the carrier of ArX as for **Set**, take $X = \mathcal{A}(ArX, \perp)$ to be the **cocarrier** of ArX, and $\forall a \in A, x \in X$ take $r(a, x) = xa$.

Every morphism $h : ArX \to BsY$ of $\mathcal{A}$ acts on $a \in A, y \in Y$ thus.

$$\mathbf{1} \xrightarrow{a} ArX \xrightarrow{h} BsY \xrightarrow{y} \perp$$

Now $y(ha) = (yh)a$, that is, $s(h(a), y) = r(a, h'(y))$ where h, h' denote respectively the left and right actions of h, making h a Chu transform.

Chu spaces are of interest because they can represent a wide range of mathematical objects, algebraic, topological, and both.

8. Qualia

The new object $\perp$ that **Chu** brings to **Set** can be understood as a Chu space in its own right having as its carrier the set K. But it can also be understood as furnishing $\mathbf{1}$ with states, a notion that is nonexistent in **Set**. Each element of the set $K = \mathcal{A}(\mathbf{1}, \perp)$ thus has the ambiguous quality of being simultaneously a covariant point and a contravariant state.

If we view points $a \in A$ as concrete entities and states $x \in X$ as mental states, this ambiguity of the elements of K would appear to provide a consistent interpretation of C.I. Lewis's notion of qualia, as having simultaneously a perceptual or psychological quality yet also being a real thing given the need for perceived objects to be real.

9. Typed Chu spaces

Unify presheaves and qualia $K : \mathbf{1} \to \perp$ by replacing $\mathbf{1}$ by base $\mathcal{J}$, $\perp$ by cobase $\mathcal{L}$, and K by a profunctor (distributor, bimodule) $\mathcal{K} : \mathcal{L} \not\to \mathcal{J}$.

$$t \xrightarrow{f} s \xrightarrow{a} A \xrightarrow{h} B \xrightarrow{x} p \xrightarrow{\varphi} q$$

- s, t Sorts in base $\mathcal{J}$.
- $f : t \to s$ The opposite of an operation symbol in $\mathcal{T} = \mathcal{J}^{op}$.
- A, B universes ("communes").
- $a : s \to A$ Element of sort s in universe A
- $h : A \to B$ Morphism in $\mathcal{A}$ transforming elements forwards and states backwards.
- $x : B \to p$ state or predicate for property p in universe B.
- p, q properties in cobase $\mathcal{L}$.
- $\varphi : p \to q$ Predicate transformer in $\mathcal{L}$ acting on p-predicates.
- $K_s^p = \mathcal{A}(s, p)$, qualia $k : s \to p$ of sort s for propery p.

10. Evolution of human consciousness

Human consciousness seems able to distinguish sorts and properties. Perhaps other animals have similar abilities, but since we cannot as yet communicate sufficiently well with them we can only speak about human consciousness.

Proposal: Thought emerges from categories as graphs with composable edges constructed at random, with randomly chosen distinguished objects $j, k, \ldots$ acting as sorts and $p, q, \ldots$ acting as properties. We then organize our comprehension of a scene as a universe possessing elements or points and states or predicates.

Natural selection then acts to favor those structures that are most helpful to survival.

While one could design a great many other ways of accomplishing the same thing, this particular approach is sufficiently simple in organization that it could well be discovered early in human evolution.