Residue Field Domination

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Introduction to Valued Fields

- Consider $\mathbb{R}(t)$, the field of rational functions with real coefficients.
- There is no $\sqrt{-1}$ so this field can be ordered. One way to do this is to say $p(t) < q(t)$ if for all sufficiently large $r \in \mathbb{R}$, $p(r) < q(r)$.
- Say $p(t) \equiv q(t)$ if $p(t) = O(q(t))$ and $q(t) = O(p(t))$. Then $R(t)^*/\equiv$ is an ordered abelian group, usually called Γ and written additively.
- The quotient map $v : R(t)^* \to \Gamma$ is called a *valuation*.
- The collection of all $p(t)$ in $R(t)$ such that $p(t) = O(1)$ is a convex ring. This is called the *valuation ring*, which we will denote V.
- The collection of all $m \in V$ such that $1/m \notin V$ forms a maximal ideal, $\mathfrak{m}$ of V. We call these elements *infinitessimals*.
- The map $\pi : V \to V/\mathfrak{m}$ is called the *standard part map*.

Valued Fields in Model Theory

- We will consider fields that are better behaved than $\mathbb{R}(t)$.
- Let R be the the real closure of $\mathbb{R}(t)$
  - that is, close $\mathbb{R}(t)$ under square roots of positive elements, and insure that polynomials of odd degree have at least one root.
- We add Γ as a sort, as well as $v : R^* \to \Gamma$.
- We add $k = V/\mathfrak{m}$ as a sort as well as $\pi : V \to k$.
- This is a *real closed valued field* and we refer to its theory as RCVF.
- If you form a field extension by adjoining a root of -1, you have an *algebraically closed valued field* and we refer to its theory as ACVF.

Some background

- Haskell, Hrushovski, and Macpherson isolated a phenomena in models of algebraically closed valued fields they called stable domination.
  - A formula, $\varphi(x, y)$, is stable if it does not have the order property.
    - i.e. there is no $(a_i b_i)_{i<\omega}$ such that $\varphi(a_i, b_j)$ iff $i < j$.
  - Thus valued fields are not stable due to the value group.
  - However, if L and M satisfy ACVF and each contain a maximal algebraically closed C with $k(L)$ algebraically independent from $k(M)$ over $k(C)$ and with $\Gamma(L) \cap \Gamma(M) = \Gamma(C)$ then $\mathrm{tp}(L/Ck(L)\Gamma(L))$ implies $\mathrm{tp}(L/M)$.
    - This (roughly) is the property called stable domination.
  - Why "However"? We need a brief detour into stability and independence relations.

What Is An Independence Relation?

- An independence relation, written $A \mathop{\smile\!\!\!\!\!\!|}^{\text{I}}_C B$, should capture the idea that B and C together contain no additional interesting information about A than C does alone.
- An example: Let $\mathfrak{M}$ be a $\mathbb{Q}$-vector space, and let V be a definable subspace of $\mathfrak{M}^2$.
  - For instance, let $\mathfrak{M} := (\mathbb{R}, +, \{q\cdot\}_{q \in \mathbb{Q}})$, and let V be the line $q_1 x + q_2 y = 0$.
- Consider two elements of $\mathbb{R}^2$, a and b, in the same coset of V. Intuitively, b should tell you more about a then you could say without parameters.
  - with the parameter b, one can say "$x - b$ is in V". This statement is true of a, and if $b_2, b_3, \ldots$ are in different cosets of V, then the formulas "$x - b_i$ is in V" define pairwise disjoint sets.
    - This is an example of "forking" and one writes $a \not\mathop{\smile\!\!\!\!\!\!|}\, b$.

A Second Example

- Let $\mathfrak{M}$ be $(\mathbb{C}, +, \cdot)$
- Consider a tuple, a, contained in $\mathbb{C}^3$ not in any algebraic surface defined over $\mathbb{Q}^{alg}$.
    - if there is a surface, defined over B, containing a then it seems reasonable to say that b has more information about a than is available over the empty set, and one would write $a \not\perp\!\!\!\perp^{I} B$
    - Assume there is no curve containing a defined over B. If there is a curve containing a defined over $C \supseteq B$, then $a \not\perp\!\!\!\perp^{I}_{B} C$
- i.e. define $A \not\perp\!\!\!\perp^{I}_{B} C$ to mean there is a tuple of elements of A which is contained over C in a variety of lower dimension than over B.
- It turns out that this is not a different independence relation. This is another example of "forking", and so we write $A \not\perp\!\!\!\perp_{B} C$.

Example

- Let $\mathscr{L} := \{E(x, y)\}$. Let T say that E is an equivalence relation with infinitely many equivalence classes, each of which is infinite.

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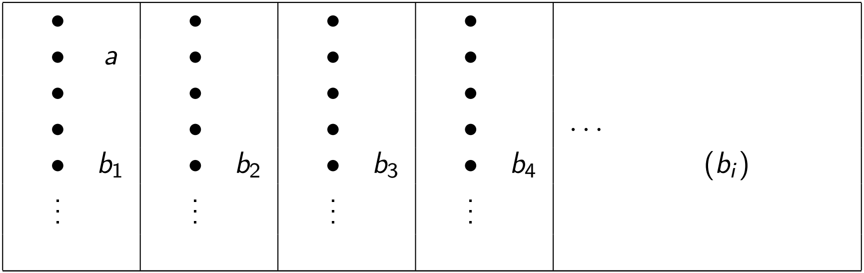
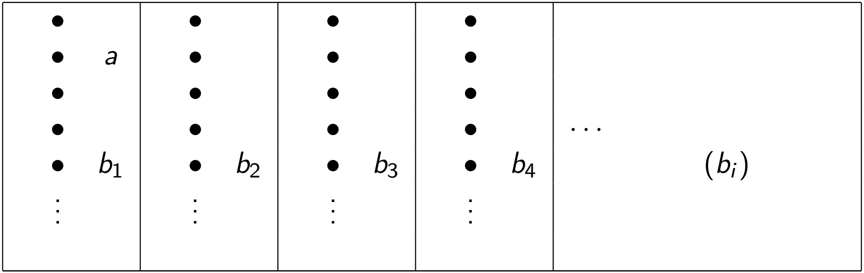


Example

- Let $\mathscr{L} := \{E(x,y)\}$. Let T say that E is an equivalence relation with infinitely many equivalence classes, each of which is infinite.
- Note that xEb_1 defines a set which includes a, and the formulas $xEb_1, xEb_2, xEb_3, \ldots$ define disjoint sets.



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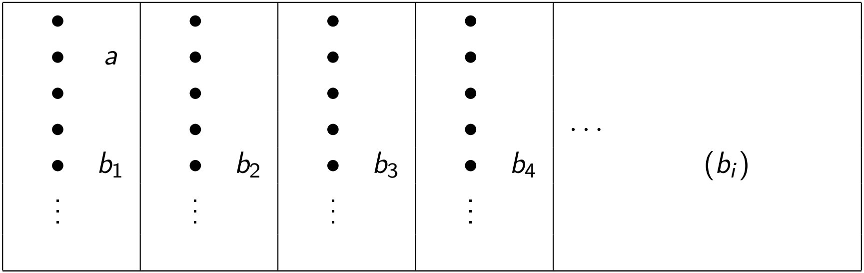

- $E(x, b_1)$ *divides* over the empty set, and $\mathrm{tp}(a/b_1)$ *forks* over the empty set.

Forking

Definition

A formula $\varphi(x, b)$ *divides* over C if there is $(b_i)_{i\in\mathbb{N}}$ such that $\{\varphi(x, b_i)|i \in \mathbb{N}\}$ is k-inconsistent, and each $b_i \in \mathrm{tp}(b/C)$.

Definition

A type *forks* over C if it implies a disjunction of formulas which divide over C.

- Each example of an independence relation so far has been non-forking.
- Non forking is written $a \mathop{\smile\!\!\!\!\!\!\mid}_b c$

Unique non-forking extensions

- Non-forking is best behaved in theories that are stable (i.e. no formula has the order property).
- Here one has, among other things, the fact that if $a \perp_C B$ and C is a model (or just algebraically closed in M^{eq}) then $\mathrm{tp}(a/C)$ implies $\mathrm{tp}(a/BC)$.
- Hence the "however" from many slides ago:
  - However, if L and M satisfy ACVF and each contain a maximal algebraically closed subfield C with $k(L)$ algebraically independent from $k(M)$ over $k(C)$ and with $\Gamma(L) \cap \Gamma(M) = \Gamma(C)$ then $\mathrm{tp}(L/Ck(L)\Gamma(L))$ implies $\mathrm{tp}(L/M)$.

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- Let $\mathscr{L} := \{<\}$. Let T be the theory of dense linear orders.
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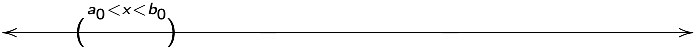

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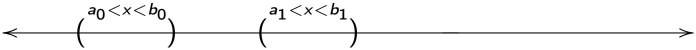

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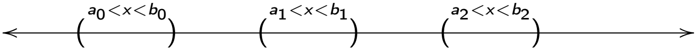

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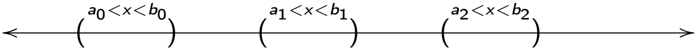

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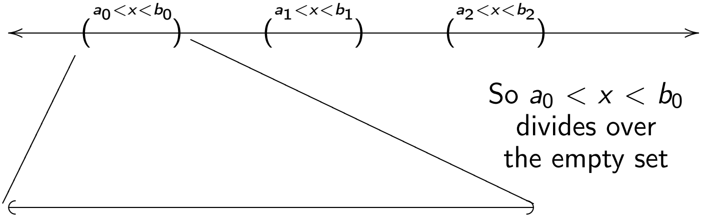

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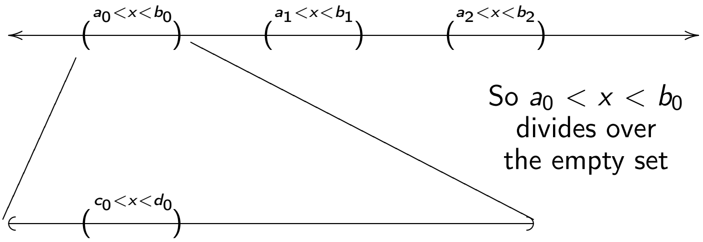

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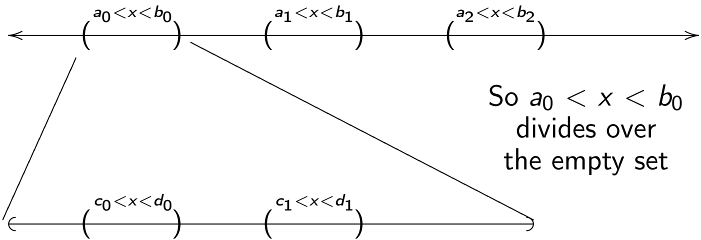

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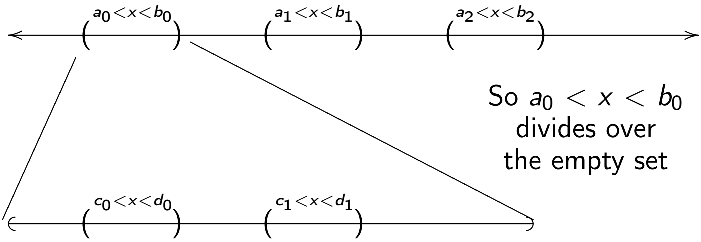

So $c_0 < x < d_0$ divides over $\{a_0, b_0\}$.

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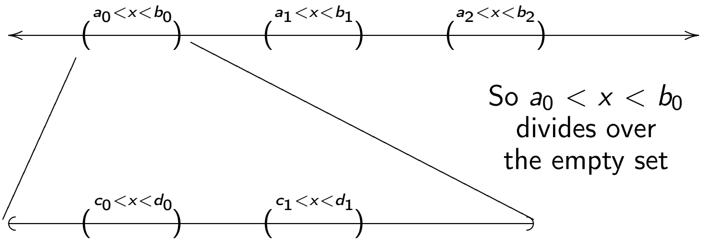

So $c_0 < x < d_0$ divides over $\{a_0, b_0\}$. There is no end to the "information" that you can have about an element. So forking is not an independence relation.

þ-Forking

- There is a generalization of stability (and of simplicity), called *rosiness* that is not ruined by the existence of an order.
- We want a definition similar to forking but that is well-behaved in a larger variety of settings.

Definition

A formula $\varphi(x, b)$ *þ-divides* over C if there is some $\theta(y, d)$ such that $\{\varphi(x, \tilde{b}) | \tilde{b} \models \theta(y, d)\}$ is k-inconsistent, and $\mathrm{tp}(b/Cd)$ is infinite and contains $\theta(y, d)$.

Definition

A type *þ-forks* over C if it implies a disjunction of formulas which þ-divide over C.

- þ-forking is a more uniform version of forking.
- One writes $a \mathop{\smile\!\!\!\!\!\!|}^{þ}_{B} C$ to indicate non-þ-forking.

Back to $(\mathbb{Q}, <)$

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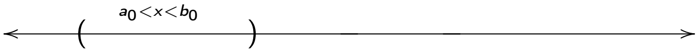

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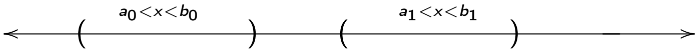

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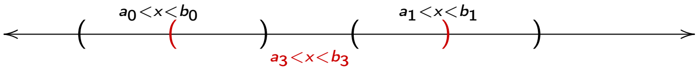

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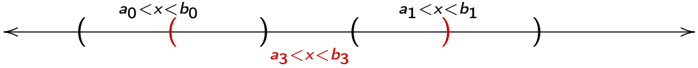

- So $a_0 < x < b_0$ does not þ-divide. (Only things of the form $x = b$ þ-divide.)

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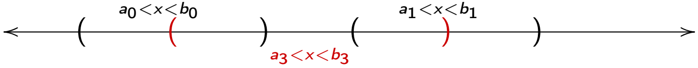

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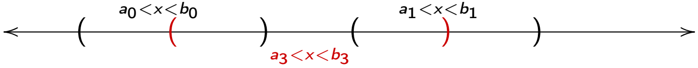

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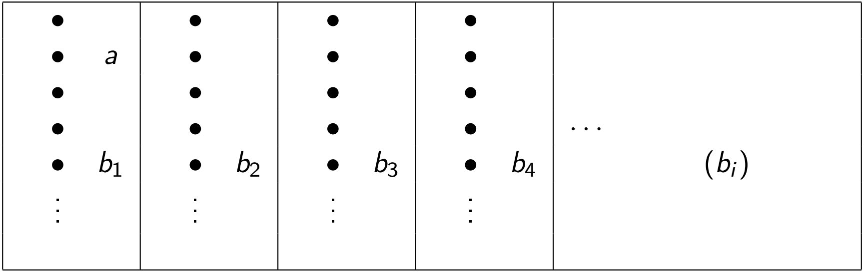
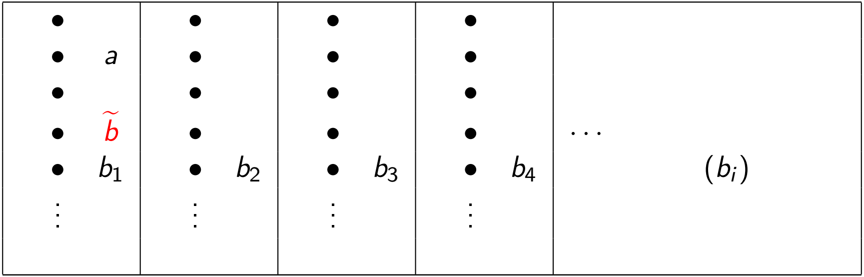
Definition

When $\mathop{\smile\hskip-0.9em ^| \ }^{þ}$ is an independence relation on $\mathfrak{M}^{eq}$, we call the theory *rosy*.

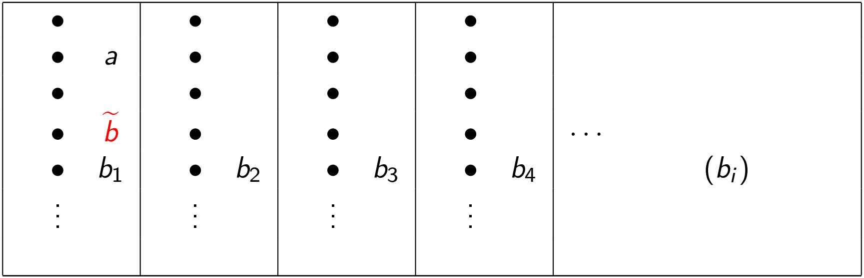
- Note in this example if $a \mathop{\smile\hskip-0.9em ^| \ }_C b$, $\mathrm{tp}(a/C)$ does not imply $\mathrm{tp}(a/Cb)$



a
b1
b2
b3
b4
. . .
(bi)



- There is no þ-dividing! xEb_1 and xEb_2 define disjoint sets, but xEb_1 and $xE\widetilde{b}$ define identical sets.



- There is no þ-dividing!
- But the problem would be solved if we could treated a/E as an element. Then the formula "x is in the equivalence class a/E" would þ-divide.
- When one adds to $\mathfrak{M}$ sorts for quotients of definable equivalence relations, one forms $\mathfrak{M}^{eq}$.
- Working in $\mathfrak{M}^{eq}$ in a stable theory, forking and þ-forking coincide.
- In any theory, non-forking is the strongest independence relation and non-þ-forking is the weakest.

Back to ACVF

- Just as forking "over reacts" to the presence of an order, þ-forking "over reacts" to the presence of an ultrametric.
- And þ-forking independence is the weakest possible independence relation, so ACVF does not admit any independence relation.
- However, when one has C, L, M with $k(L) \mathop{\smile\!\!\!\!\!\!\mid}_C k(M)$ and $\Gamma(L) \mathop{\smile\!\!\!\!\!\!\mid}^{þ}_C \Gamma(M)$ then when C is maximal and algebraically closed, $\mathrm{tp}(L/Ck(L)\Gamma(L))$ implies $\mathrm{tp}(L/M)$.
- Philosophy: Once one controls for the value group, ACVF is one stable structure sitting on top of another one.

Residue Field Domination

- Idea: After accounting for the value group, a real closed valued field is an o-minimal structure sitting on top of another o-minimal structure.
- Guess: If $C \models RCVF$ be a maximal field which is a submodel of both L and M, and suppose that $k(L)\Gamma(L) \mathop{\smile\hskip-0.9em ^| \ }^{\text{þ}}_{C} k(M)\Gamma(M)$ then $\mathrm{tp}(L/Ck(L)\Gamma(L))$ together with $\mathrm{tp}_{<}(L/M)$ implies $\mathrm{tp}(L/M)$.

Theorem (E., Haskell, Maříková)

In fact, $\mathrm{tp}(L/Ck(L)\Gamma(L))$ *implies* $\mathrm{tp}(L/M)$.

Theorem (E., Haskell, Maříková)

In either RCVF or ACVF. Suppose C is maximal and a model. Then

i) $a \mathop{\smile\hskip-0.9em ^| \ }^{\text{þ}}_{C} b$ *if and only if* $k(Ca)\Gamma(Ca) \mathop{\smile\hskip-0.9em ^| \ }^{\text{þ}}_{C} k(Cb)\Gamma(Cb)$,

ii) $a \mathop{\smile\hskip-0.9em ^| \ }_{C} b$ *if and only if* $k(Ca)\Gamma(Ca) \mathop{\smile\hskip-0.9em ^| \ }_{C} k(Cb)\Gamma(Cb)$.