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Hausdorff gave an abstract definition of space by means of neighborhood systems of points of the space. Brouwer started developing grounds for rejecting classical reasoning in favor of constructive reasoning. Lewis suggested to resolve the paradoxes of material implication by introducing strict implication. This resulted in a number of logical systems, fourth of which will play a prominent role in our story.
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Kuratowski gave the first pointfree definition of a topological space by means of a closure operator on the powerset. Alexandroff gave another, now widely accepted, pointfree definition of a topological space by means of open sets. Several attempts were made to analyze carefully Brouwer’s new logic (Kolmogorov, Glivenko, Heyting).
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1930's:
Gödel defined a translation of intuitionistic logic into modal logic, which allowed to view intuitionistic logic as a fragment of S4 (Lewis' fourth system).
Stone and Tarski gave a topological representation of algebras associated with intuitionistic logic. This resulted in Tarski's topological interpretation of intuitionistic logic.

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McKinsey and Tarski introduced closure algebras as an algebraic language for topological spaces. They proved that every closure algebra can be represented as a subalgebra of the powerset algebra equipped with topological closure. This resulted in topological interpretation of modal logic. As a consequence of the two representation theorems, they proved that Gödel's translation is full and faithful.
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Algebras of topology

- Alexandroff way: $X \mapsto \Omega(X) = \text{the algebra of all opens of } X$.
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Closure algebras

McKinsey-Tarski (1944):

A closure algebra is a pair $(B, c)$, where $B$ is a Boolean algebra and $c : B \to B$ satisfies Kuratowski's axioms:

1. $c(0) = 0$
2. $a \leq c(a)$
3. $cc(a) \leq c(a)$
4. $c(a \lor b) = c(a) \lor c(b)$

Let $i : B \to B$ be the interior operator dual to $c$, that is, $i(a) = -c(-a)$.

Then $H := \{i(a) : a \in B\}$ is a Heyting algebra.

Heyting algebra = bounded distributive lattice in which $\land$ has residual $\to$:

$a \land x \leq b$ iff $x \leq a \to b$.
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Thus, the open elements of a closure algebra form a Heyting algebra. Is every Heyting algebra represented this way? The answer is yes, and this is at the heart of seeing that the Gödel translation is full and faithful.

To see this, it is convenient to first discuss representation of closure algebras and Heyting algebras. These representations generalize the celebrated Stone representation of Boolean algebras.
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To see this, it is convenient to first discuss representation of closure algebras and Heyting algebras. These representations generalize the celebrated Stone representation of Boolean algebras.
For a Boolean algebra $B$, let $X := \{\text{ultrafilters of } B\}$ be the ultrafilters of $B$. Define $\beta : B \rightarrow \mathcal{P}(X)$ by $\beta(a) = \{x \in X \mid a \in x\}$.

Then $\beta : B \rightarrow \mathcal{P}(\mathcal{B})$ is a Boolean embedding. Moreover, $\{\beta(a) \mid a \in B\}$ is a basis of a Stone topology (compact Hausdorff zero-dimensional topology) on $X$. Furthermore, $B$ is isomorphic to the Boolean algebra of clopens ($=\text{closed and open sets}$) of this topology.
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McKinsey-Tarski topology

McKinsey and Tarski weakened the Stone topology by weakening the basis to
\{β(i)a | a ∈ B\}
We call the weaker topology the McKinsey-Tarski topology.

Key Lemma:
β(i)a = \text{int}\ β(a)
where \text{int} is the interior in the McKinsey-Tarski topology.

McKinsey-Tarski representation: Every closure algebra can be represented as a subalgebra of the closure algebra (℘(X), cl) for some topological space X.
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Consequently, every Heyting algebra can be represented as a subalgebra of the Heyting algebra of opens of some topological space.
From Heyting algebras to closure algebras

We can now see that each Heyting algebra can be realized as the opens of a closure algebra.

Construction:
Let $H$ be a Heyting algebra.
By the Stone-Tarski theorem, represent $H$ as a subalgebra of the opens $\Omega(X)$ of a topological space $X$.
Then $\Omega(X)$ is the open elements of the closure algebra $(\mathcal{P}(X), cl)$.
Let $(B, c)$ be the subalgebra of $(\mathcal{P}(X), cl)$ generated by $H$.
Then $H$ is precisely the opens of $(B, c)$.
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Gödel's translation of the intuitionistic language $\text{IL}$ into the modal language $\text{ML}$ associates with each formula $\phi$ of $\text{IL}$ the formula $\phi^t$ of $\text{ML}$ obtained by prefixing $□$ to each subformula of $\phi$.

Intuition: Think of $\phi$ as an element of the Lindenbaum algebra $\mathcal{H}$ of intuitionistic logic. Since $\mathcal{H}$ is a Heyting algebra, each element of $\mathcal{H}$ can be thought of as an open element of an appropriate closure algebra $(\mathcal{B}, c)$. Thus, $\phi$ gets interpreted in $(\mathcal{B}, c)$ as $\phi^t$.

Gödel-McKinsey-Tarski Theorem: $\text{IPC} \vdash \phi$ iff $\text{S4} \vdash \phi^t$. 
Gödel translation

Gödel’s translation of the intuitionistic language $\mathcal{IL}$ into the modal language $\mathcal{ML}$ associates with each formula $\varphi$ of $\mathcal{IL}$ the formula $\varphi^t$ of $\mathcal{ML}$ obtained by prefixing $\Box$ to each subformula of $\varphi$. 

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Gödel translation

Gödel’s translation of the intuitionistic language $\mathcal{IL}$ into the modal language $\mathcal{ML}$ associates with each formula $\varphi$ of $\mathcal{IL}$ the formula $\varphi^t$ of $\mathcal{ML}$ obtained by prefixing $\Box$ to each subformula of $\varphi$.

**Intuition:** Think of $\varphi$ as an element of the Lindenbaum algebra $\mathcal{H}$ of intuitionistic logic. Since $\mathcal{H}$ is a Heyting algebra, each element of $\mathcal{H}$ can be thought of as an open element of an appropriate closure algebra $(B, c)$. Thus, $\varphi$ gets interpreted in $(B, c)$ as $\varphi^t$.

**Gödel-McKinsey-Tarski Theorem:** $\text{IPC} \vdash \varphi$ iff $\text{S4} \vdash \varphi^t$. 
Every non-theorem of $S4$ can be refuted in the closure algebra of the real line, and every non-theorem of $IPC$ can be refuted in the Heyting algebra of opens of the real line. More generally, the real line can be replaced by an arbitrary crowded separable metric space (for example, an Euclidean space, the rational line, or Cantor's discontinuum). Rasiowa and Sikorski showed that separable can be dropped from the assumptions.
McKinsey-Tarski completeness

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Rasiowa and Sikorski showed that separable can be dropped from the assumptions.
Let \((B, c)\) be a closure algebra, \(H\) be the Heyting algebra of open elements, and \(X\) be the set of ultrafilters of \((B, c)\).

Define a binary relation \(R\) on \(X\) by \(xRy\) iff \(x \cap H \subseteq y\). Then \(R\) is a preorder (reflexive and transitive), and it is a partial order iff \(B\) is generated as a Boolean algebra by \(H\).

Since \(R\) is a preorder, it gives rise to the Alexandroff topology \(\tau_R\) on \(X\), where the closure of \(U \subseteq X\) is given by \(R^{-1}[U] = \{x \in X | \exists u \in U \text{ with } xRu\}\).

Key Lemma: \(\beta(c) = R^{-1}[\beta(a)]\).

Jónsson-Tarski (1951), Kripke (1963): Every closure algebra can be represented as a subalgebra of the closure algebra \((\mathcal{P}(X), R^{-1})\) for some preordered set \((X, R)\).
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Key Lemma: \(\beta(ca) = R^{-1}[\beta(a)]\).
Let \((B, \mathfrak{c})\) be a closure algebra, \(H\) be the Heyting algebra of open elements, and \(X\) be the set of ultrafilters of \((B, \mathfrak{c})\). Define a binary relation \(R\) on \(X\) by

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**Key Lemma:** \(\beta(\mathfrak{c}a) = R^{-1}[\beta(a)]\).

**Jónsson-Tarski (1951), Kripke (1963):** Every closure algebra can be represented as a subalgebra of the closure algebra \((\wp(X), R^{-1})\) for some preordered set \((X, R)\).
The three topologies

1. The Stone topology $\tau_S$ with clopen basis $\{\beta(a) | a \in B\}$.

2. The McKinsey-Tarski topology $\tau_{MT}$ with open basis $\{\beta(ia) | a \in B\}$.

3. The Alexandroff topology $\tau_R$ of the preorder $R$.

Theorem:

1. $\tau_{MT} = \tau_S \cap \tau_R$.

2. $R$ is the specialization preorder of $\tau_{MT}$ (that is, $x R y$ iff $x$ belongs to the McKinsey-Tarski closure of $y$).
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Next phase of the program

Dummett-Lemmon (1959): The correspondence $\text{IPC} \rightarrow \text{S4}$ can be extended to extensions of $\text{IPC}$ and $\text{S4}$. For an extension $L$ of $\text{IPC}$, let $M = \text{S4} + \{ \varphi | L \vdash \varphi \}$. Then $L \vdash \psi$ iff $M \vdash \psi$. The logic $M$ is referred to as a modal companion of $L$.

For an extension $M$ of $\text{S4}$, let $L = \text{IPC} + \{ \varphi | M \vdash \varphi \}$. Then $L \vdash \psi$ iff $M \vdash \psi$. The logic $L$ is referred to as the intuitionistic fragment of $M$.

Extensions of $\text{S4}$ have unique intuitionistic fragments, but extensions of $\text{IPC}$ have many modal companions.
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Extensions of $\text{S4}$ have unique intuitionistic fragments, but extensions of $\text{IPC}$ have many modal companions.
Grzegorczyk logic

In 1968 Grzegorczyk introduced a new modal companion of IPC, which turned out to be of fundamental importance. S4 is the logic of all closure algebras. The Grzegorczyk logic \( \text{Grz} \) is the logic of those closure algebras \((B, c)\) in which \(B\) is generated as a Boolean algebra by the Heyting algebra \(H\) of open elements of \((B, c)\).

Esakia (1976): \( \text{Grz} \) is the largest modal companion of IPC.

Therefore, an extension \(M\) of S4 is a modal companion of IPC iff \( S4 \subseteq M \subseteq \text{Grz} \).

The Blok-Esakia theorem (1976): The lattice of extensions of IPC is isomorphic to the lattice of extensions of \( \text{Grz} \).
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The Blok-Esakia theorem (1976): The lattice of extensions of $\text{IPC}$ is isomorphic to the lattice of extensions of $\text{Grz}$. 
Further directions

One of the consequences of the McKinsey-Tarski completeness theorem is that many important properties of topological spaces are not expressible in the language of closure algebras. For example, we cannot tell apart the real line from Cantor's discontinuum or Euclidean spaces of dimension greater than 1.

One option to increase expressivity is to work with derivative instead of closure.

\[ x \in \text{cl}(A) \iff U_x \cap A \neq \emptyset \text{ for every open neighborhood } U_x \text{ of } x. \]

\[ x \in \text{d}(A) \iff (U_x \setminus \{x\}) \cap A \neq \emptyset \text{ for every open neighborhood } U_x \text{ of } x. \]

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$$\text{cl}(A) = A \cup \text{d}(A)$$
Derivational logics

Working with derivative yields the concept of a derivative algebra \((B, d)\). The correspondence between Heyting algebras and closure algebras can be extended to include derivative algebras by setting \(c a = a \lor d a\). The logic of derivative algebras is the weak \(K4\). \(wK4\) is the logic of all topological spaces when \(\Box\) is interpreted as derivative. \(K4\) is the logic of all \(Td\)-spaces (the derivative of a set is closed). Derivational logics can express the \(T0\)-separation axiom, but cannot express higher separation axioms.
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- Derivational logics can express the \(T0\)-separation axiom, but cannot express higher separation axioms.
Derivational logics can distinguish between the real line, Cantor’s discontinuum, and Euclidean spaces of dimension $> 1$. But they cannot distinguish between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n, m > 1$. Gödel’s celebrated incompleteness theorem is expressible in derivational logic ($\neg \square \bot \rightarrow \neg \square \neg \square \bot$). The Gödel-Löb logic is the logic of scattered spaces. The expressive power can be increased further by adding the universal modality. This, for example, allows to express whether a space is connected. But there are other topological properties (for example, being Hausdorff, that it cannot express). The expressive power can be further extended by introducing nominals. But this may lead to undecidability of our system. One direction of current research is to seek a good balance between expressive power and decidability of a modal system.
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Gödel’s celebrated incompleteness theorem is expressible in derivational logic ($\neg \Box \bot \rightarrow \neg \Box \neg \Box \bot$). The Gödel-Löb logic is the logic of scattered spaces. The expressive power can be increased further by adding the universal modality. This, for example, allows to express whether a space is connected. But there are other topological properties (for example, being Hausdorff, that it cannot express). The expressive power can be further extended by introducing nominals. But this may lead to undecidability of our system. One direction of current research is to seek a good balance between expressive power and decidability of a modal system.
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A closer analysis of the McKinsey-Tarski theorem shows that to refute non-theorems of $S_4$ it is sufficient to work with Borel sets on the real line. Therefore, $S_4$ is the logic of the closure algebra $(\text{Bor}(\mathbb{R}), \text{cl})$. This yields the following natural question: Which extensions of $S_4$ can be picked up as logics of subalgebras of the closure algebra $(\text{P}(\mathbb{R}), \text{cl})$? Of course, instead of the real line, we can consider any space for which the McKinsey-Tarski theorem is applicable. Surprisingly, this question has a positive solution. For example, we can pick up every extension of $S_4$ from subalgebras of the closure algebra of Cantor’s discontinuum! This is no longer so if we work with the real line (connectedness gets in the way). Nevertheless, it is possible to describe the logics that arise as logics of subalgebras of the closure algebra of the real line.
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The Lebesgue measure algebra $\mathcal{M}$ is obtained from $\mathcal{B}(\mathbb{R})$ by modding out Borel sets of measure zero.

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First-order logics

The Gödel translation of IPC into S4 extends to the predicate case. However, it remains an open problem whether there is a predicate analogue of the Blok-Esakia theorem. One of the difficulties is the lack of adequate semantics in the predicate case. While both first-order intuitionistic logic and first-order S4 are complete (algebraically, topologically, or relationally), this is no longer true for many extensions of these logics. Thus, it is desirable to obtain a workable adequate semantics of these systems. Some attempts in this direction include sheaf semantics, and more generally, bundle semantics. We do have an adequate semantics for one-variable fragments of these systems by means of monadic Heyting algebras and monadic modal algebras. But in its general form, the Blok-Esakia theorem remains unsolved even for these weaker systems (some partial results in this direction are available).
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Conclusion

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