Tarski lectures 2024
UC Berkeley

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Plan of talks

Lecture I: Closures and flows in real tori: a model theoretic approach

Lecture II: The interplay of o-minimality and discrete groups

Lecture III: From closures to Hausdorff limits, in tori and nilmanifolds

Lecture I

Closures and flows in real tori: a model theoretic approach

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The closure problem

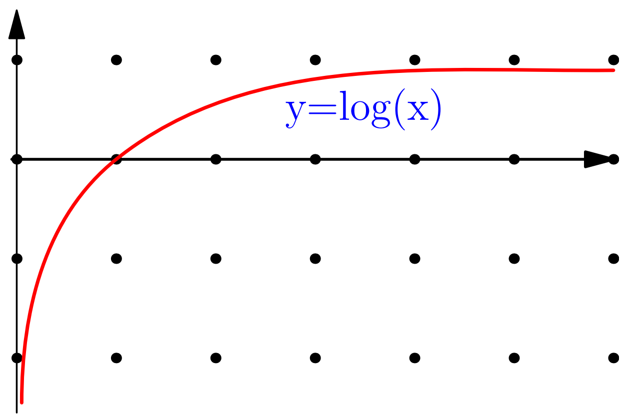
The general question

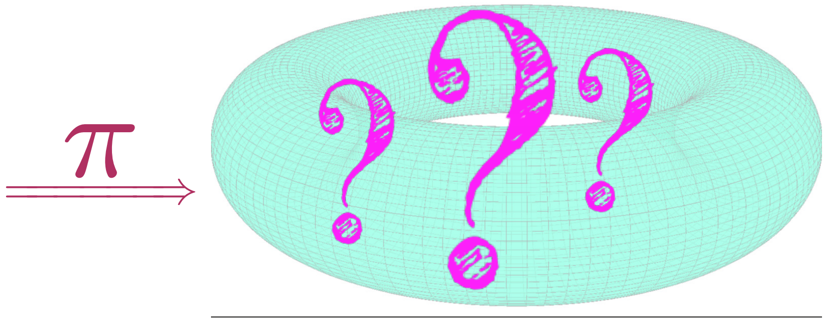
Let $\mathbb{T}$ be an abelian variety or a compact torus.

For $K = \mathbb{R}$ or $K = \mathbb{C}$, let $\pi : K^n \to \mathbb{T}$ be the covering map.

Given a “tame” set $X \subseteq K^n$ (e.g. semialgebraic).

What is the closure of $\pi(X)$ in $\mathbb{T}$?





Some history: Ax, Lindemann, Weierstrass

Theorem (Lindemann (1882)-Weierstrass (1885))

Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ be algebraic numbers. If $\alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly independent then $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are algebraically independent over $\mathbb{Q}$.

Theorem (Ax (1972))

Let $X \subseteq \mathbb{C}^n$ be an irreducible complex algebraic variety and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}[X]$ regular functions on X. If $\alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ then $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are algebraically independent over $\mathbb{C}$.

Theorem (Ax-Lindemann for complex tori, geometric version)

Let $\mathbb{T}$ be a compact complex torus and $\pi : \mathbb{C}^n \to \mathbb{T}$ be a covering map. If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety then the complex analytic Zariski closure of $\pi(X)$ in $\mathbb{T}$ is a translate of a complex subtorus of $\mathbb{T}$

From Zariski closure to topological closure

Theorem (Ax-Lindemann for complex tori, geometric version)

Let $\mathbb{T}$ be a compact complex torus and $\pi : \mathbb{C}^n \to \mathbb{T}$ be a covering map. If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety then the complex analytic Zariski closure of $\pi(X)$ in $\mathbb{T}$ is a translate of a complex subtorus of $\mathbb{T}$

Ullmo-Yafaev, 2015

What can be said about the topological closure of $\pi(X)$ in $\mathbb{T}$?

Does a version of Ax-Lindemann Theorem hold for it?

When $X \subseteq \mathbb{C}^n$ is an algebraic **curve**, Ullmo and Yafaev described the closure $\pi(X)$ in terms of cosets of real subtori of A.

Placing the problem in $\mathbb{R}^n$

Even for $X \subseteq \mathbb{C}^n$ algebraic, the closure of $\pi(X)$ brings-in real tori, hence the problem fits better into the real (not complex) setting:

- Let $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$ be a lattice in $\mathbb{R}^n$, i.e. Λ is a subgroup generated by a basis $(\omega_1, \ldots, \omega_n)$ of $\mathbb{R}^n$. Let $\mathbb{T}_\Lambda$ be the quotient group $\mathbb{R}^n/\Lambda$.
- The group $\mathbb{T}_\Lambda$ is called an n-dimensional (real) torus, and admits the structure of a compact Lie group.
- Let $\pi_\Lambda : \mathbb{R}^n \to \mathbb{T}$ be the quotient map. It is a smooth group homomorphism and $\ker(\pi_\Lambda) = \Lambda$.

Reformulating the problem

Given a "tame" set $X \subseteq \mathbb{R}^n$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what can be said about the topological closure of $\pi_\Lambda(X)$ in $\mathbb{T}_\Lambda$?

Tameness and o-minimality

For the rest of the talks, we take “tame” to mean o-minimal.

Recall that the following structures are o-minimal:

$\langle \mathbb{R}; <, +, \cdot \rangle$ (Tarski)

The definable sets are semi-algebraic. E.g. solutions to $p(\bar{x}) > 0$, for $p(\bar{x}) \in \mathbb{R}[\bar{x}]$.

$\mathbb{R}_{\text{exp}} = \langle \mathbb{R}; <, +, \cdot, e^x \rangle$ (Wilkie)

E.g. solutions to $\exists x\, p(e^x, e^{e^y}, x, y, z) > 0$, for $p(\bar{x}) \in \mathbb{R}[\bar{x}]$.

$\mathbb{R}_{\text{an,exp}} = \langle \mathbb{R}_{\text{exp}}, (f \restriction [0,1]^n)_{f \in \mathcal{F}} \rangle$ (van den Dries - Miller)

The expansion of $\mathbb{R}_{\text{exp}}$ by all restricted real analytic functions. E.g. solutions to $\forall z\ \arctan(e^{\sin x} - y^2 + 3z) > 0$, for $x \in [-1, 1], y \in \mathbb{R}$.

O-minimality and closure

From now on, we fix an o-minimal structure $\mathbb{R}_{\rm om} = \langle \mathbb{R}; <, +, \cdot, \cdots \rangle$.

The o-minimal formulation

Given a definable set $X \subseteq \mathbb{R}^n$ in $\mathbb{R}_{\rm om}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what can we say about the topological closure of $\pi_\Lambda(X)$ in $\mathbb{T}_\Lambda$?

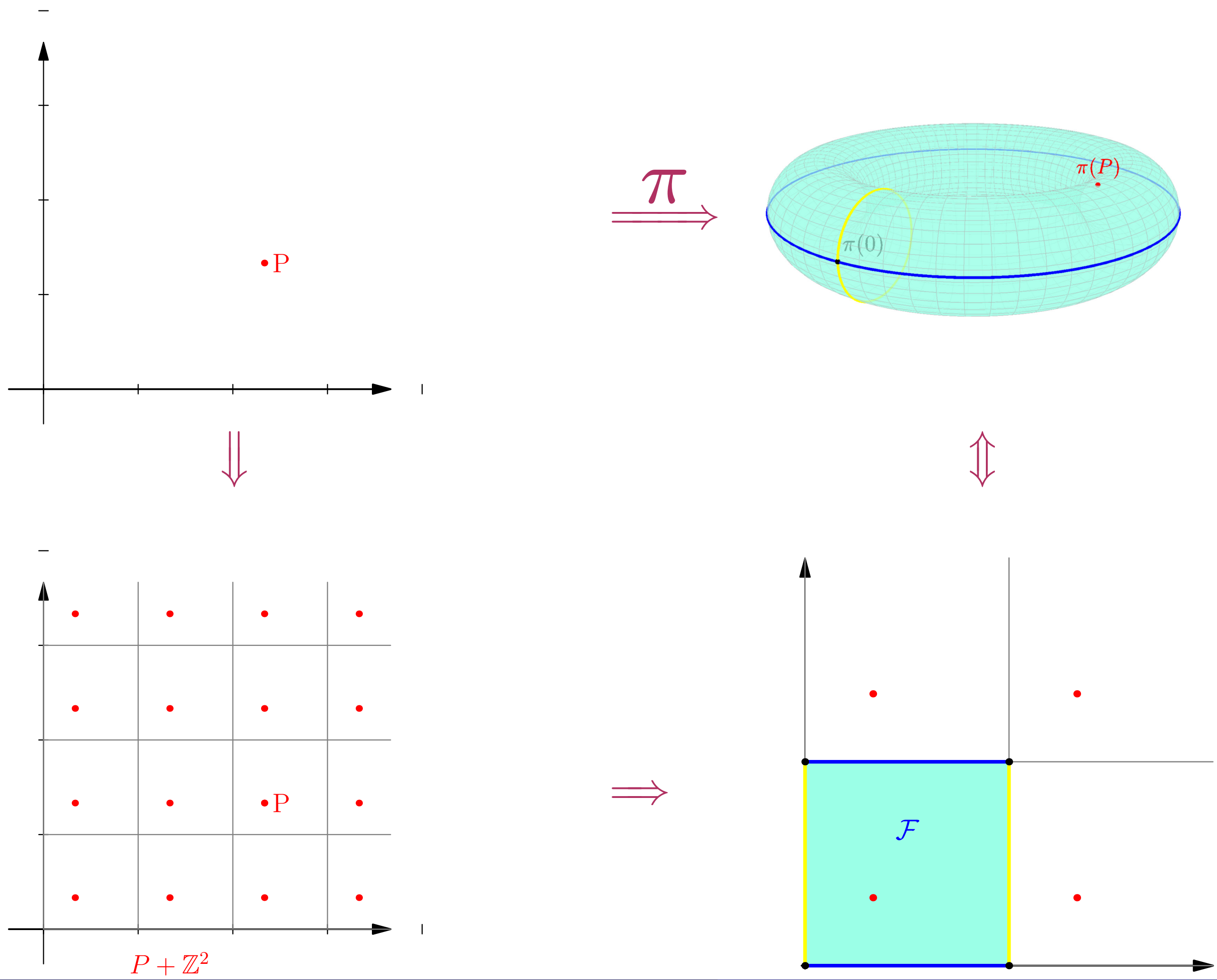
When the setting is clear we use π and $\mathbb{T}$ instead of π_Λ and $\mathbb{T}_\Lambda$.

Observation

For any $X \subseteq \mathbb{R}^n$ we have $\mathrm{cl}(\pi(X)) = \pi(\mathrm{cl}(X + \Lambda))$.

So, from now one we work with $\mathrm{cl}(X + \Lambda)$ in $\mathbb{R}^n$ (instead of $\mathrm{cl}(\pi(X))$).

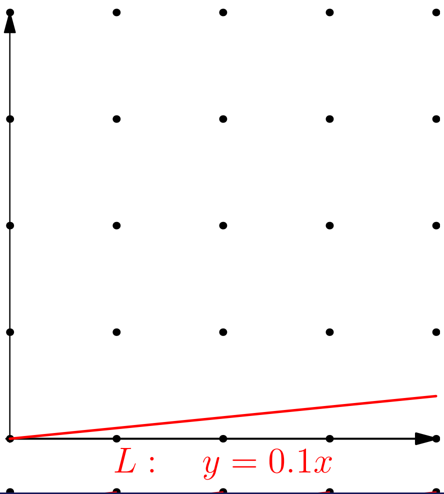
X vs $X + \Lambda$ vs $\pi({}_\Lambda(X)$



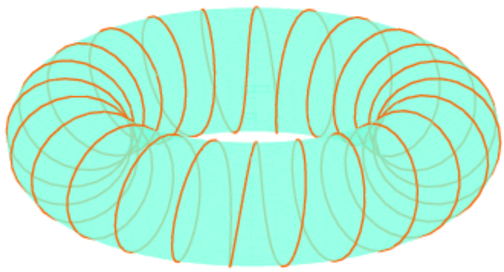
Examples

An important example: linear spaces

- Assume that $L \subseteq \mathbb{R}^n$ is an $\mathbb{R}$-subspace.
- Then $\mathrm{cl}(L + \Lambda)$ is a real Lie subgroup of $\mathbb{R}^n$
- Its connected component is an $\mathbb{R}$-subspace, with a basis in Λ. Denote it by L^Λ. It is the smallest $\mathbb{R}$-subspace of $\mathbb{R}^n$, containing L with a basis in Λ and $\mathrm{cl}(L + \Lambda) = L^\Lambda + \Lambda$.
- $\pi(L^\Lambda)$ is a (closed) real subtorus of $\mathbb{T}$.

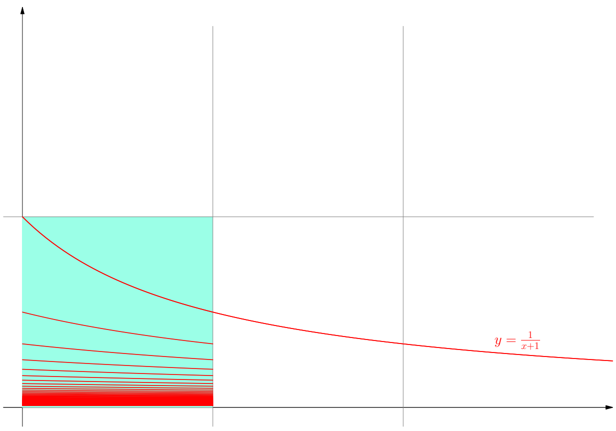


$\Longrightarrow$



Some Examples: Curves on Tori

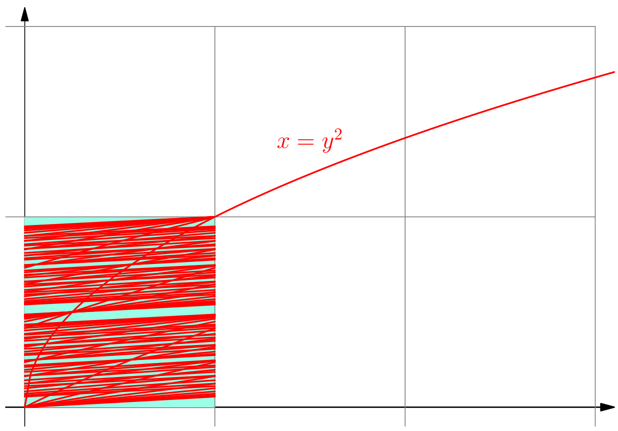
Let X be the semialgebraic curve $y = \dfrac{1}{x+1}, \quad x \geqslant 0$.
We translate it to the fundamental domain by elements of $\mathbb{Z}^2$



We have $\mathrm{cl}(X + \mathbb{Z}^2) = \Big(X \cup (x\text{-axis})\Big) + \mathbb{Z}^2$.

Some Examples: Curves on Tori

Let X be the semialgebraic curve $x = y^2, \quad y \geqslant 0$.
We translate it to the fundamental domain by elements of $\mathbb{Z}^2$.



after 100 translates

We have $\mathrm{cl}(X + \mathbb{Z}^2) = \mathbb{R}^2$.

A uniform closure theorem

Recall thal if $L \subseteq \mathbb{R}^n$ is a linear subspace and $\Lambda \subseteq \mathbb{R}^n$ is a lattice then L^Λ is the smallest linear subspace containing L with a basis in Λ.

Theorem

Let $X \subseteq \mathbb{R}^n$ be a closed definable set in $\mathbb{R}_{\text{om}}$. Then there are $\mathbb{R}$-subspaces $L_1, \dots, L_k \subseteq \mathbb{R}^n$, and definable closed sets $C_1, \dots, C_k \subseteq \mathbb{R}^n$ such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mathrm{cl}_{\mathbb{R}^n}(X + \Lambda) = \left[X \cup \bigcup_{i=1}^{k} (L_i^\Lambda + C_i)\right] + \Lambda.$$

I.e.

$$\mathrm{cl}_{\mathbb{T}_\Lambda}(\pi_\Lambda(X)) = \pi_\Lambda(X) \cup \bigcup_{i=1}^{k} (T_i + \pi_\Lambda(C_i)),$$

where $T_i = \pi(L_i^\Lambda)$ are real subtori of $\mathbb{T}_\Lambda$.

A model theoretic approach

- Let $\mathbb{R}_{full}$ be the expansion of $\mathbb{R}$ by all subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$.

  And let $\mathcal{R} \succ \mathbb{R}_{full}$ be an $|\mathbb{R}|^+$-saturated elementary extension.
- For $X \subseteq \mathbb{R}^n$, let $X^\sharp$ denote its realization in $\mathcal{R}$.
- Let $\mathcal{O} = \{\alpha \in \mathcal{R} : \exists r \in \mathbb{R}\ |\alpha| < r\}$, the ring of finite elements.
- Let $\mu = \{\epsilon \in \mathcal{R} : \forall r \in \mathbb{R}\ |\epsilon| < r\}$, the infinitesimals, a maximal ideal in $\mathcal{O}$. We have $\mathcal{O} = \mathbb{R} \oplus \mu$.
- Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ denote the standard part map (also, the residue map). Namely, $\mathrm{st}(\alpha) =$ the unique $r \in \mathbb{R}$ such that $\alpha \in r + \mu$.

  We extend it coordinate-wise to $\mathrm{st} : \mathcal{O}^n \to \mathbb{R}^n$.
- For $Y \subseteq \mathcal{R}^n$, let $\mathrm{st}(Y) := \mathrm{st}(\mathcal{O}^n \cap Y)$.

Closure through the standard part map

A key observation

If $Y \subseteq \mathbb{R}^n$ then $\mathrm{cl}(Y) = \mathrm{st}(Y^\sharp)$.

Proof

- Assume $a \in \mathrm{cl}(Y)$.
  Then $B(a,r) \cap Y \neq \varnothing$ for every $r \in \mathbb{R}^{>0}$.
  By saturation, there is $\alpha \in \bigcap_{r \in \mathbb{R}^{>0}} B^\sharp(a,r) \cap Y^\sharp$.
  Obviously $\mathrm{st}(\alpha) = a$. Hence $\mathrm{cl}(Y) \subseteq \mathrm{st}(Y^\sharp)$.
- Assume $a = \mathrm{st}(\alpha)$, for $\alpha \in Y^\sharp$. Then $|a - \alpha| \in \mu$, hence $Y^\sharp \cap B(a,r) \neq \varnothing$ for every $r \in \mathbb{R}^{>0}$. Thus, the same is true in $\mathbb{R}_{full}$, so $a \in \mathrm{cl}(Y)$.

The non-standard formulation of the question

For $X \subseteq \mathbb{R}^n$ definable in the o-minimal structure $\mathbb{R}_{\mathrm{om}}$ and for a lattice $\Lambda \subseteq \mathbb{R}^n$, what is $\mathrm{st}(X^\sharp + \Lambda^\sharp) \subseteq \mathbb{R}^n$?

Complete o-minimal types appear

We have $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{\text{om}}$.

Partition into types

Let $S_X(\mathbb{R})$ be the collection of all complete $\mathbb{R}_{\text{om}}$-types over $\mathbb{R}$, on X (i.e. containing the formula $x \in X$).

For $p(x) \in S_X(\mathbb{R})$, we let $p(\mathcal{R})$ be its set of realizations in $\mathcal{R}$.

We have:

$$\operatorname{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \operatorname{st}(p(\mathcal{R}) + \Lambda^\sharp).$$

The new question

For a complete type $p \in S_X(\mathbb{R})$, what is $\operatorname{st}(p(\mathcal{R}) + \Lambda^\sharp)$?

Tarski Lecture II

The interplay of o-minimality and discrete subgroups

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April 24, 2024

Recalling the problem

Fix $\mathbb{R}_{\mathrm{om}}$ an o-minimal expansion of the real field.

The closure problem

Given a definable set $X \subseteq \mathbb{R}^n$ in $\mathbb{R}_{\mathrm{om}}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what is the topological closure of $\pi(X)$ in $\mathbb{T} = \mathbb{R}^n/\Lambda$?

Equivalently, what is the closure of $X + \Lambda$ in $\mathbb{R}^n$?

We fixed $\mathcal{R} \succ \mathbb{R}_{full}$ which is $|\mathbb{R}|^+$-saturated. For $S \subseteq \mathbb{R}^n$ let $S^\sharp = S(\mathcal{R})$.

Let $\mathcal{O} = \{\alpha \in \mathcal{R} : \exists r \in \mathbb{R}\ |\alpha| < r\}$, the ring of finite elements,
and $\mu = \{\epsilon \in \mathcal{R} : \forall r \in \mathbb{R}\ |\epsilon| < r\}$, the ideal of infinitesimals.

We have $\mathcal{O} = \mathbb{R} \oplus \mu$ and denote by $\mathrm{st}\colon \mathcal{O} \to \mathbb{R}$ the "standard part map".

Fact. For $S \subseteq \mathbb{R}^n$ we have $\mathrm{cl}(S) = \mathrm{st}(S^\sharp) := \mathrm{st}(S^\sharp \cap \mathcal{O}^n)$.

The non-standard formulation

What is $\mathrm{st}(X^\sharp + \Lambda^\sharp) \subseteq \mathbb{R}^n$?

Reducing the problem

We also have, $\mathrm{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$.

Localizing the problem

For a complete o-minimal type $p \in S_n(\mathbb{R})$ and a lattice $\Lambda \subseteq \mathbb{R}^n$ describe the set $\mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$.

Remark

For any set $Y \subseteq \mathcal{R}^n$ we have $\mathrm{st}(Y) = \mathrm{st}(\mu + Y)$.

Theorem (Λ-linearity of types)

For any complete o-minimal type $p \in S_n(\mathbb{R})$ there are $a_p \in \mathbb{R}^n$ and a linear subspace $L_p \subseteq \mathbb{R}^n$ such that for any lattice $\Lambda \subseteq \mathbb{R}^n$ we have

$$\mu + p(\mathcal{R}) + \Lambda^\sharp = \mu + a_p + L_p^\sharp + \Lambda^\sharp.$$

O-minimality vs. discrete subgroups

Recall

A structure on $\mathbb{R}$ is o-minimal if every definable subset of $\mathbb{R}$ is a finite union of intervals with end points in $\mathbb{R} \cup \{\pm\infty\}$.

In particular, every definable discrete subset of $\mathbb{R}^n$ is finite.

Thus, a lattice $\Lambda \subseteq \mathbb{R}^n$, and $\pi : \mathbb{R}^n \to \mathbb{T} = \mathbb{R}^n/\Lambda$ are **not** definable in any o-minimal structure.

So, in general, the set $X + \Lambda$ is not definable in any o-minimal structure.

How can we use o-minimality?

"Linearize" $\mu + p(\mathcal{R})$ independently of Λ.

μ-stabilizers of o-minimal types play major role.

For simplicity, we mostly consider one-dimensional types, i.e. types on o-minimal curves at ∞.

O-minimal detour I: one dimensional types

Let $\gamma : (0, \infty) \to \mathbb{R}^n$ be an $\mathbb{R}_{\mathrm{om}}$-definable curve.

For any $\mathbb{R}_{\mathrm{om}}$-definable set $X \subseteq \mathbb{R}^n$ exactly one of the sets $\{t \in (0, \infty) : \gamma(t) \in X\}$ for $\{t \in (0, \infty) : \gamma(t) \in \neg X\}$ is unbounded.

Thus there is a unique complete $\mathbb{R}_{\mathrm{om}}$-type over $\mathbb{R}$, containing all sets $\{\gamma(t) : t > r\}$, for $r \in \mathbb{R}$.

Linearizing $\mu + p(\mathcal{R})$: first step

For $p \in S_n(\mathbb{R})$, we let

$$\mathrm{Stab}_\mu(p) = \{g \in \mathbb{R}^n : g + \mu + p(\mathcal{R}) = \mu + p(\mathcal{R})\},$$

and call it the μ-stabilizer of p.

Theorem (2015)

1. $\mathrm{Stab}_\mu(p)$ is a definable subgroup (linear subspace) of $\mathbb{R}^n$.
2. If p is unbounded then $\dim(\mathrm{Stab}_\mu(p)) > 0$.

An analogue holds for **any** definable group in $\mathbb{R}_{\mathrm{om}}$.

The intuition

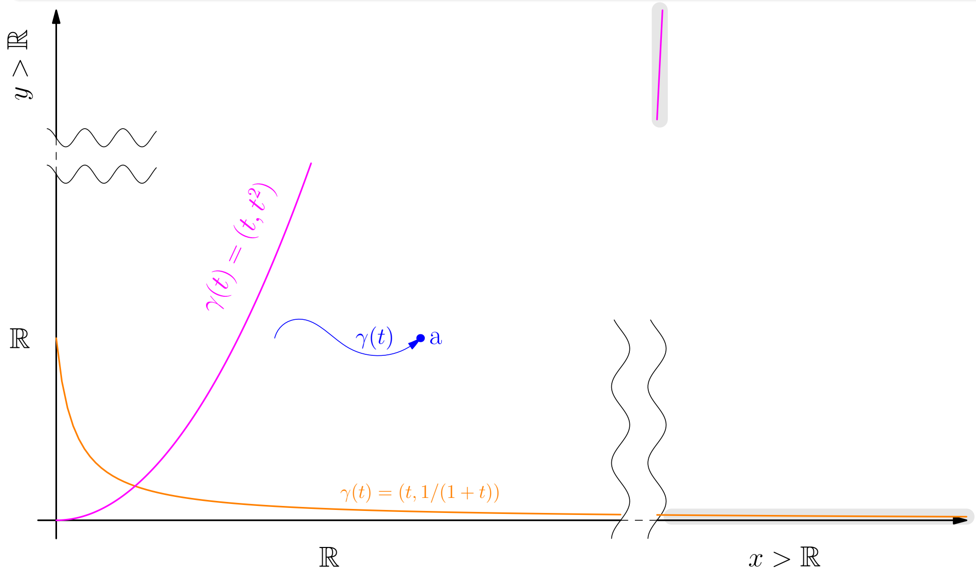
Unbounded o-minimal types are almost “flat”.

Example

Example

Below p is the type on a curve γ at ∞

- When $\lim_{t\to\infty}\gamma(t)=a$, for $a\in\mathbb{R}^2$, then $\mathrm{Stab}_\mu(p)=\{0\}$.
- When $\gamma(t)=(t,1/(t+1))$, then $\mathrm{Stab}_\mu(p)=\mathbb{R}\times\{0\}$.
- When $\gamma(t)=(t,t^2)$, then $\mathrm{Stab}_\mu(p)=\{0\}\times\mathbb{R}$.



Linearizing $\mu + p(\mathcal{R})$: The nearest coset

For a type $p \in S_n(\mathbb{R})$, consider all affine subspaces, $A = a + L \subseteq \mathbb{R}^n$ (L a linear subspace) defined over $\mathbb{R}$, such that $p(\mathcal{R}) \subseteq \mu + A^\sharp$.

Definition+Fact

The intersection A_p of all the above affine spaces is itself an affine space defined over $\mathbb{R}$, and $p(\mathcal{R}) \subseteq \mu + A_p^\sharp$. We call it the nearest coset to p and denote by A_p.

Note

For a type $p \in S_n(\mathbb{R})$, A_p is invariant under $\mathrm{Stab}_\mu(p)$.

- Indeed, if $g \in \mathrm{Stab}_\mu(p)$, $g + \mu + p(\mathcal{R}) = \mu + p(\mathcal{R})$.
- Since $p(\mathcal{R}) \subseteq \mu + A_p^\sharp$, we have

$$\mu + p(\mathcal{R}) = g + \mu + p(\mathcal{R}) \subseteq g + \mu + A_p^\sharp = \mu + (g + A_p)^\sharp.$$

- From the minimality of A_p, we conclude $g + A_p = A_p$. □

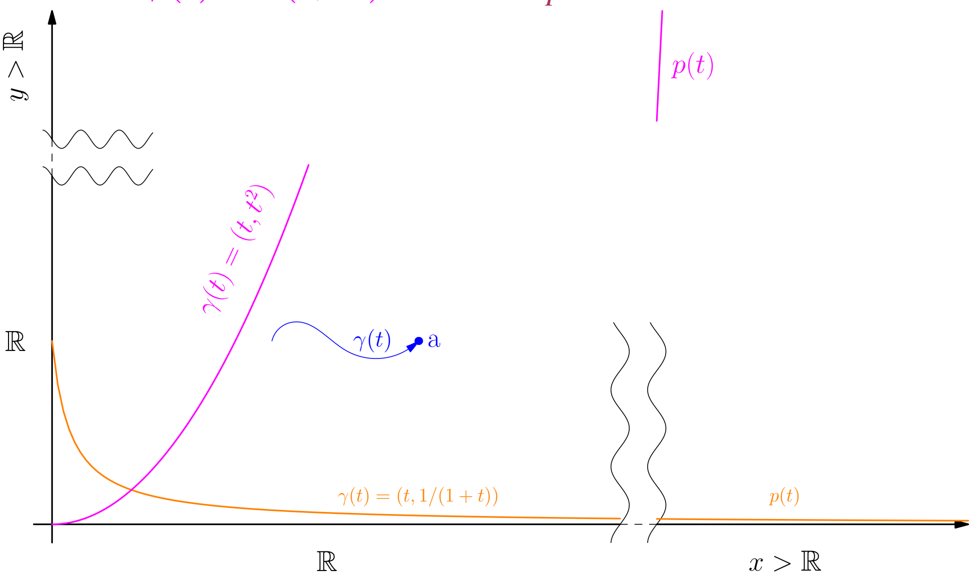
Example:

Below p is the type on a curve γ at ∞

When $\lim_{t\to\infty}\gamma(t) = a$, for $a \in \mathbb{R}^2$, then $A_p = \{0\}$.

When $\gamma(t) = (t, 1/(t+1))$, then $A_p = \mathbb{R} \times \{a\}$.

When $\gamma(t) = (t, t^2)$, then $A_p = \mathbb{R}^2$.



Notice: for $\alpha \models p$ we have $\alpha + \mathrm{Stab}_\mu(p) \subseteq \mu + p(\mathcal{R}) \subseteq \mu + A_p^\sharp$.

O-minimal types are linear mod lattices

Theorem (Λ-linearity of types)

Let $p \in S_n(\mathbb{R})$, with $A_p = a_p + L_p$. Then for every lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mu + p(\mathcal{R}) + \Lambda^\sharp = \mu + a_p + L_p^\sharp + \Lambda^\sharp.$$

Proof.

- If $p(\mathcal{R})$ is bounded, i.e. $p(\mathcal{R}) \subseteq a + \mu$ for some $a \in \mathbb{R}^n$, then $A_p = \{a\}$ and the theorem follows.
- Asume $p(\mathcal{R})$ is unbounded. Then $\mathrm{Stab}_\mu(p)$ is infinite, and both sides are invariant under $\mathrm{Stab}_\mu(p)$.
- Use factorization by $\mathrm{Stab}_\mu(p)^\Lambda$ to reduce dimension.

□

Back to the closure theorem

Corollary (First Main Step)

If $X \subseteq \mathbb{R}^n$ is definable then for any lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mathrm{cl}(X+\Lambda) = \mathrm{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp) =$$

$$\bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(a_p + L_p^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{cl}(a_p + L_p + \Lambda) =$$

$$\bigcup_{p \in S_X(\mathbb{R})} (a_p + L_p^\Lambda + \Lambda).$$

Next step:

Simplify

$$\bigcup_{p \in S_X(\mathbb{R})} (a_p + L_p^\Lambda + \Lambda).$$

The definability of the collection of nearest cosets

Let $X \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}_{\text{om}}$. Applying the theory of Tame Pairs (v.d. Dries), we obtain

Theorem

The family of nearest cosets $\{A_p : p \in S_X(\mathbb{R})\}$ is definable in $\mathbb{R}_{\text{om}}$.

Corollary (Definability)

There is a definable family of affine subspaces of $\mathbb{R}^n$, $\{a_t + L_t : t \in T\}$ (depending only on X) such that for any lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mathrm{cl}(X + \Lambda) = \bigcup_{t \in T} (a_t + L_t^{\Lambda} + \Lambda).$$

The closure theorem

Using Baire Category Theorem and o-minimal cell decomposition, we conclude:

Theorem

Let $X \subseteq \mathbb{R}^n$ *be closed, definable in* $\mathbb{R}_{\text{om}}$.
Then there are infinite $\mathbb{R}$*-subspaces* $L_1, \ldots, L_r \subseteq \mathbb{R}^n$*, and definable closed sets* $C_1, \ldots, C_r \subseteq \mathbb{R}^n$ *such that for every lattice* $\Lambda \subseteq \mathbb{R}^n$

$$\text{cl}_{\mathbb{R}^n}(X + \Lambda) = \left[X \cup \bigcup_{i=1}^{r} (L_i^\Lambda + C_i) \right] + \Lambda.$$

Hence for $\pi : \mathbb{R}^n \to \mathbb{T} = \mathbb{R}^n/\Lambda$ *we have*

$$\text{cl}_{\mathbb{T}}(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^{r} (T_i + \pi(C_i)),$$

where $T_i = \pi(L_i^\Lambda)$ *are real subtori of* $\mathbb{T}$.

A curious corollary

Corollary

Given $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{\mathrm{om}}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, there is an $\mathbb{R}_{\mathrm{om}}$-definable set $Y_\Lambda \subseteq \mathbb{R}^n$, such that

$$\mathrm{cl}(\pi(X)) = \pi(Y_\Lambda).$$

Addendum: on equidistribution

Let $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ be a torus and $\pi \colon \mathbb{R}^n \to \mathbb{T}$ the projection map.

Let $\gamma(t) \colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^n$ be a definable curve in $\mathbb{R}_{\mathrm{om}}$, and for $R \geqslant 0$ let $\gamma_R = \gamma \cap B(0,R)$.

For $X \subseteq \mathbb{T}$ let $\mu_{\gamma,R}(X) = \dfrac{\text{length of } (\gamma_R \cap \pi^{-1}(X))}{\text{length of } \gamma_R}$.

Each $\mu_{\gamma,R}$ is a probability measure on $\mathbb{T}$.

Theorem (P-S, Ulmo-Yafaev for semialgebraic curves)

Assume $\mathbb{R}_{\mathrm{om}}$ is polynomially bounded.
Then $\mathrm{cl}_{\mathbb{T}}(\pi(\gamma)) = \mathbb{T}$ ***if and only if***

$$\lim_{R\to\infty} \mu_{\gamma,R} = \mu_{\mathbb{T}},$$

where $\mu_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$.

Namely, $\pi(\gamma)$ is dense in $\mathbb{T}$ iff it is "equidistributed" in $\mathbb{T}$.

On Equidistribution

Equidistribution fails in general o-minimal structures.

Example

Let $\gamma(t) \colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^2$ be given by $x = t$, $y = e^t$.
Then $\pi(\gamma)$ is dense in $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$.

But, the family of measures $\mu_{\gamma,R}$ does not converge (as R goes to infinity).

Tarski Lecture III

From closure to Hausdorff limits in tori (and nilmanifolds)

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April 26, 2024

Generalizing the closure problem

In the first two talks we discussed:

The closure problem

Given $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what is $\mathrm{cl}(\pi(X))$ in $\mathbb{T} = \mathbb{R}^n/\Lambda$?
The answer used linear spaces associated to complete types over $\mathbb{R}$, on X.

We want to extend the result in two directions:

From closure to Hausdorff limits

(From tori to nilmanifolds)

Generalizing the closure problem

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The answer used linear spaces associated to complete types over $\mathbb{R}$, on X.

We want to extend the result in two directions:

From closure to Hausdorff limits

(From tori to nilmanifolds)

The Hausdofff limits of a definable family

Question

Let $\{X_t : t \in T\}$ *be a family of subsets of* $\mathbb{R}^n$ *definable in an o-minimal structure on* $\mathbb{R}$*. For a lattice* $\Lambda \subseteq \mathbb{R}^n$*, we consider the possible Hausdorff limits of the family* $\{\pi_\Lambda(X_t) : t \in T\}$ *in* $\mathbb{T}_\Lambda$*.*

When are some (or all) Hausdorff limits equal to $\mathbb{T}_\Lambda$*?*

Hausdorff distance and limit

Definition

Given a metric space (M, d), and $X, Y \subseteq M$,

$$d_H(X, Y) = \inf\{\epsilon > 0 : X \subseteq Y^\epsilon \ Y \subseteq X^\epsilon\}, \text{where}$$
$$Y^\epsilon = \{x \in M : d(x, Y) \leqslant \epsilon\}.$$

We have $d_H(X, Y) = 0 \Leftrightarrow \mathrm{cl}(X) = \mathrm{cl}(Y)$.
Also, d_H is a metric on the collection of compact subsets of M.

Definition

Given a family $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ of relatively compact subsets of M, we say that a compact set $Y \subseteq M$ is a Hausdorff limit at ∞ of $\mathcal{F}$ if there is an unbounded sequence $t_n \in (0, \infty)$, such that $\lim_{n\to\infty} d_H(\mathrm{cl}(X_{t_n}), Y) = 0$.

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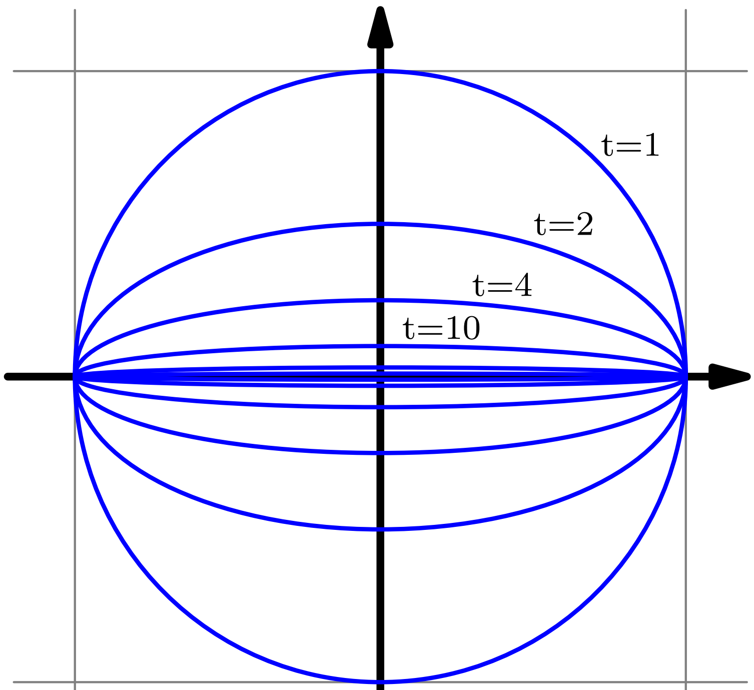
An Example

A family of ellipses

$$\mathcal{F}: \; X_t = \{(x, y) : x^2 + (ty)^2 = 1\}, \;\; t \in [1, , \infty).$$

The (unique) Hausdroff limit at ∞ is the interval $[-1, 1] \times \{0\}$.



The new question

A question (A. Nevo)

Assume that $\mathcal{F} = \{X_t : t \in (0,\infty)\}$ is a definable family of subsets of $\mathbb{R}^n$ in an o-minimal structure, and $\Lambda \subseteq \mathbb{R}^n$ a lattice,
Describe the family of Hausdorff limits of $\pi_\Lambda(\mathcal{F}) := \{\pi_\Lambda(X_t) : t \in (0,\infty)\}$ at ∞ inside $\mathbb{T}_\Lambda$.
In particular, when is $\mathbb{T}_\Lambda$ the unique Hausdorff limit at ∞, of $\pi_\Lambda(\mathcal{F})$?

Notice that the closure problem is a special case of the above (for $X \subseteq \mathbb{R}^N$, consider the constant family $X_t = X$, for all $t \in (0,\infty)$.

As in the closure problem, we may study the problem inside the fundamental domain $F_\Lambda \subseteq \mathbb{R}^n$.

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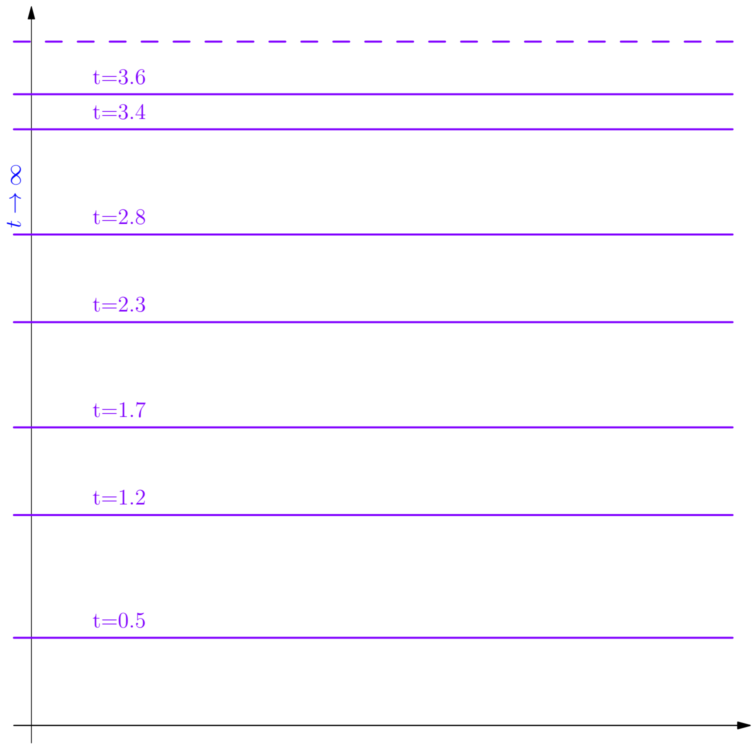
Horizontal lines in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Let $\mathcal{F} : X_t = \mathbb{R} \times \{t\}$, $t \in (0, \infty)$.



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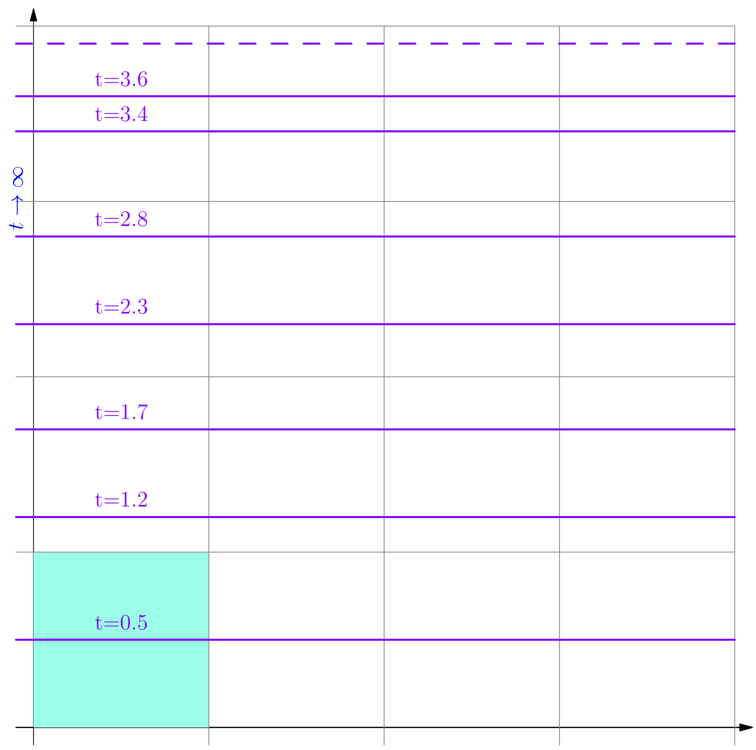
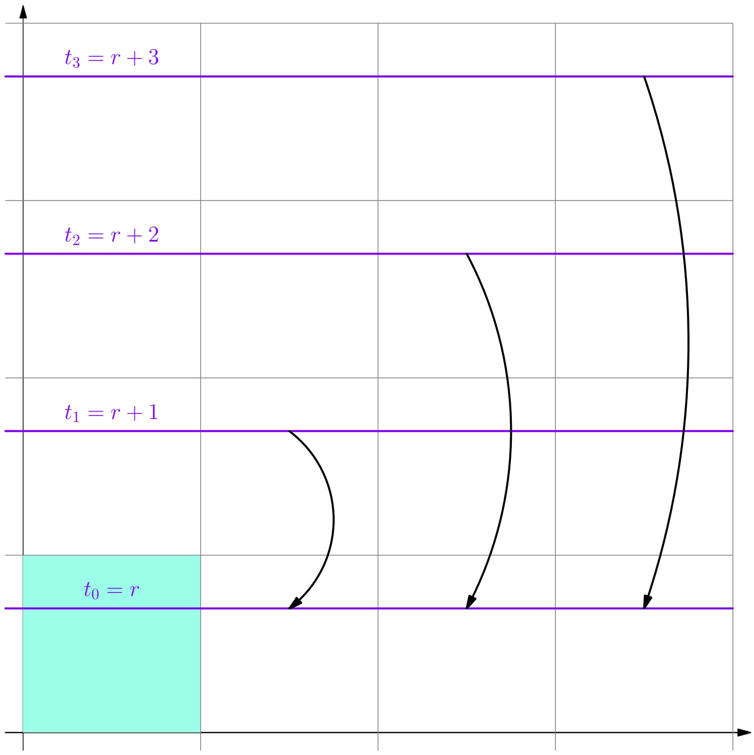


Consider $\pi_\Lambda(\mathcal{F}) = \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$.

Horizontal lines in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

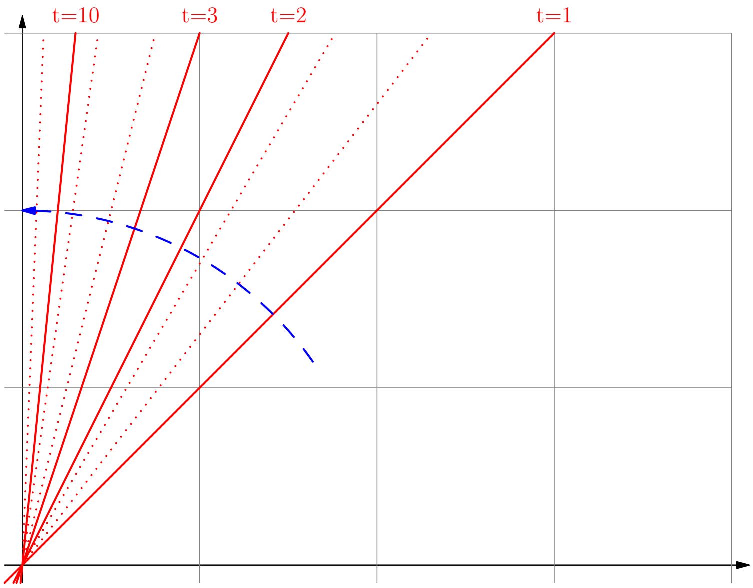
For $L = \mathbb{R} \times \{0\}$, the Hausdorff limits at ∞ are exactly the cosets of $\pi_\Lambda(L)$ in $\mathbb{T}_\Lambda$.



For each $r \in \mathbb{R}$, the sequence $(\pi_\Lambda(X_{r+n}))_{n=0}^{\infty}$ is constant.

Lines of increasing slope in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Consider the family $\mathcal{F} : X_t = \{(x, tx) : t \in (0, \infty)\}$.



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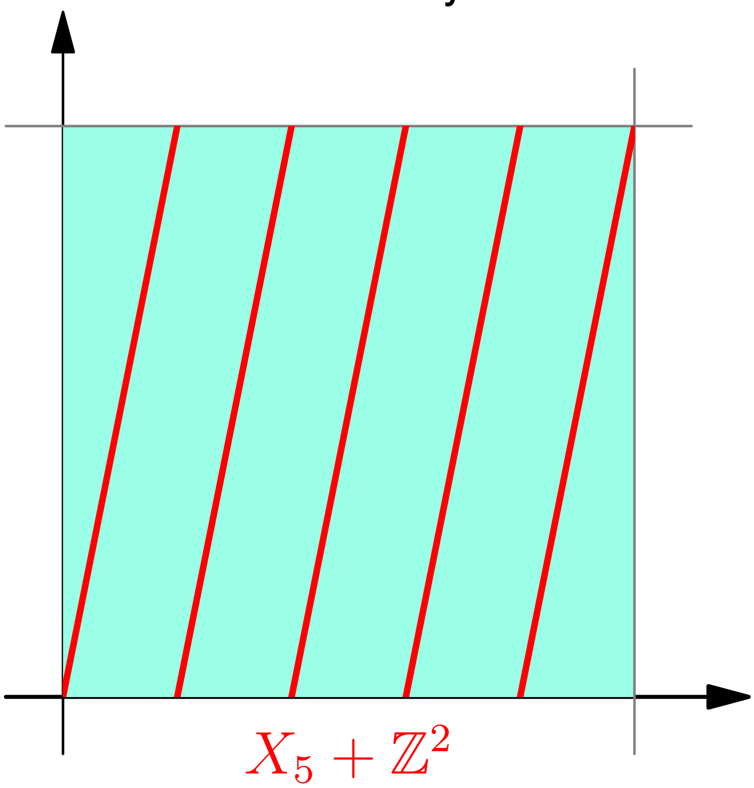
The (unique) Hausdorff limit of $\{\pi_\Lambda(X_t) : t \in (\infty)\}$ at ∞ is $\mathbb{T}_\Lambda$. This remains true for every lattice.



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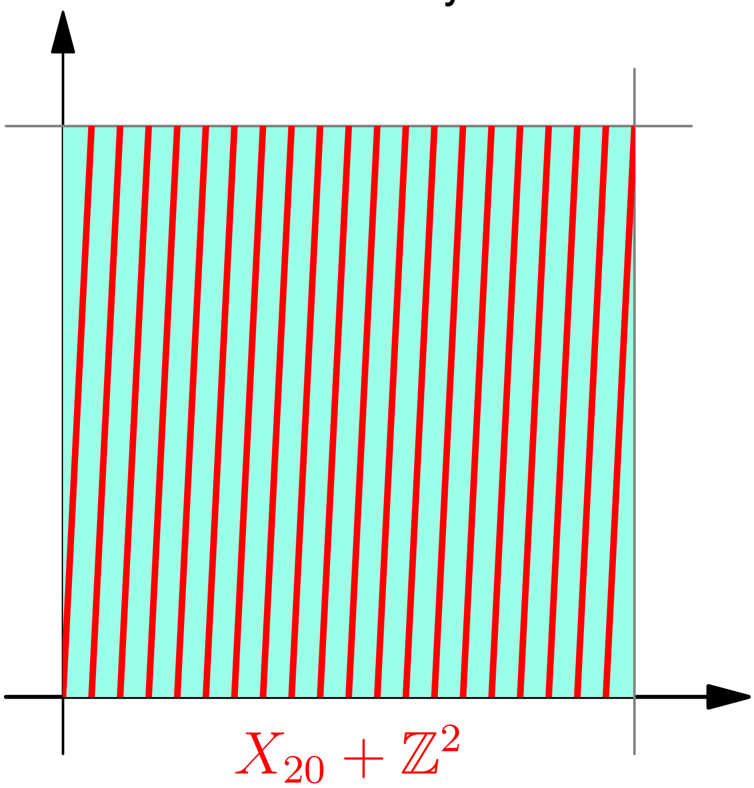

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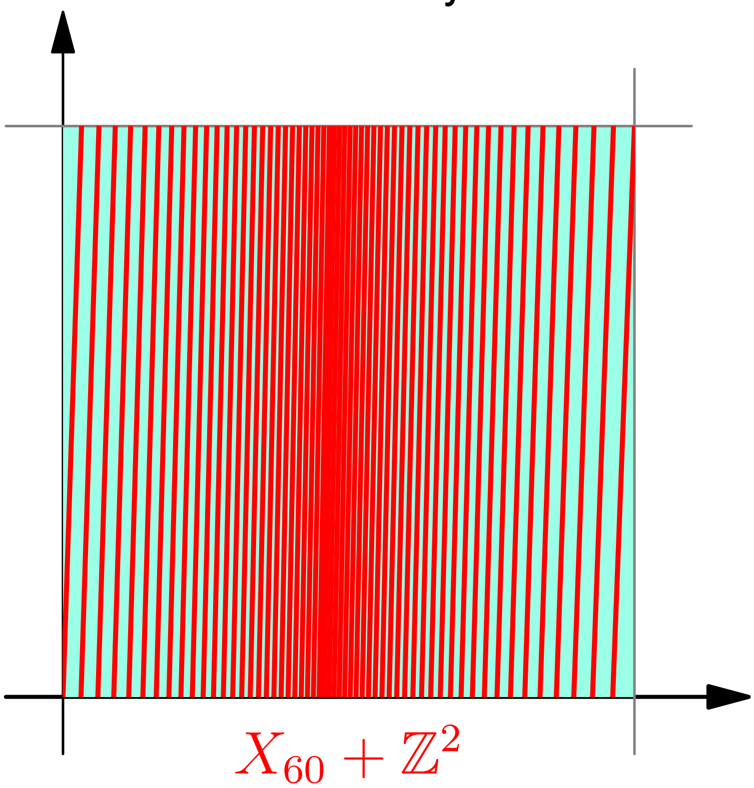
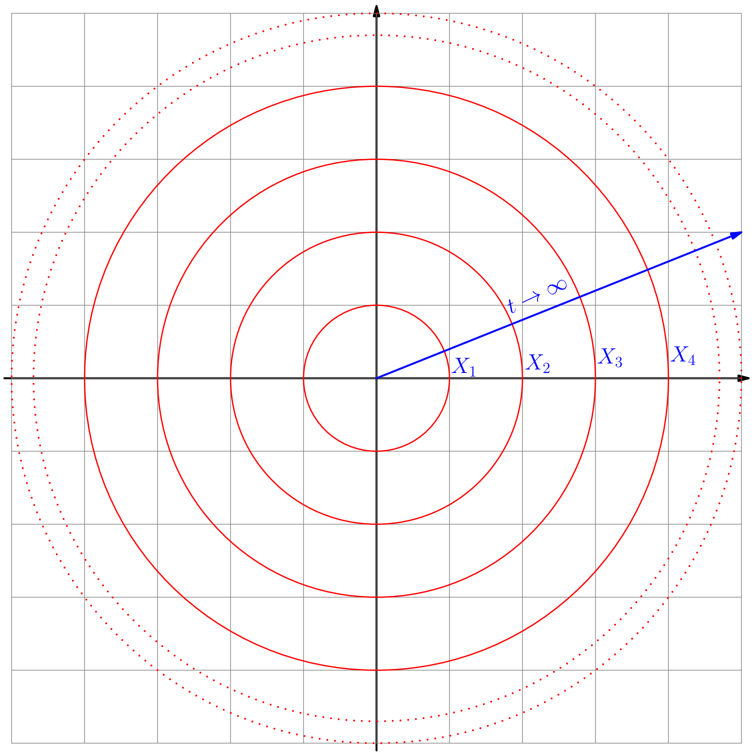


Circles of increasing radius in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Consider the family $X_t = \{(x, y) : x^2 + y^2 = t^2\}$, for $t \in (0, \infty)$.



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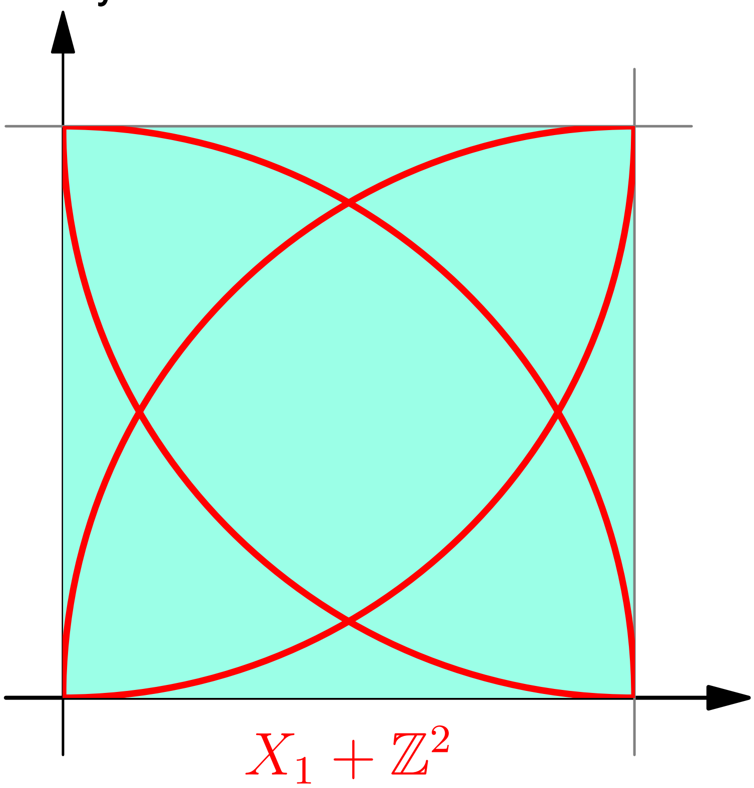

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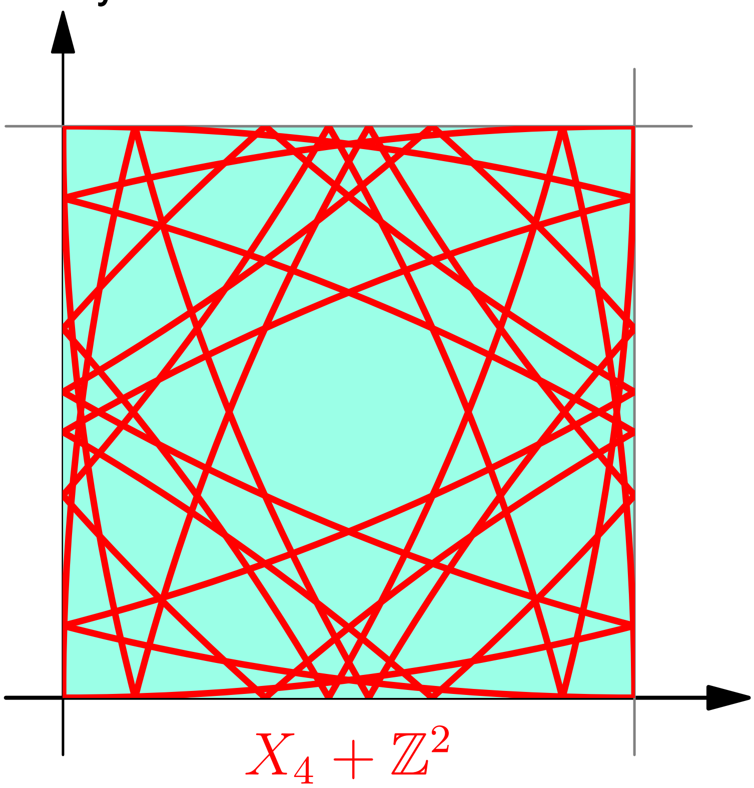

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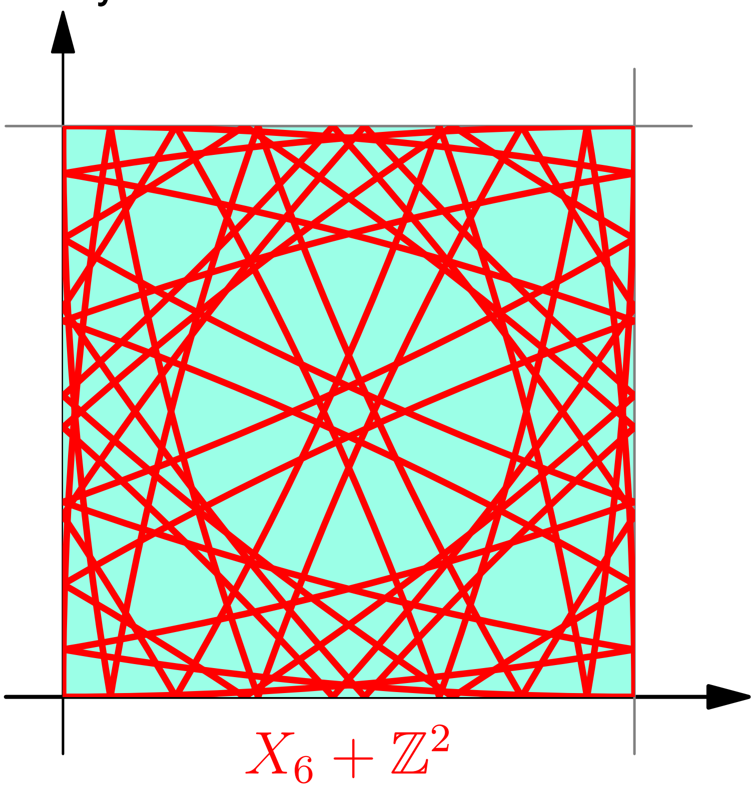

A model theoretic approach

We let $\mathcal{R} \succ \mathbb{R}_{full}$, $\mathcal{O} \subseteq \mathcal{R}$ the ring of finite elements, μ the ideal of infinitesimals in $\mathcal{O}$ and $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ the standard part map.
For $S \subseteq \mathbb{R}^n$, we let $S^\sharp = S(\mathcal{R})$ and $\mathrm{st}(S^\sharp) = \mathrm{st}(S^\sharp \cap \mathcal{O}^n)$

Fact (based on L. Narens, 1972)

Let $\{X_t : t \in (0, \infty)\}$ be a family of subsets of $\mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^n$ a lattice. For a compact $Y \subseteq \mathbb{T}_\Lambda$, the following are equivalent

1. *Y is a Hausdorff limit at ∞ of $\{\pi_\Lambda(X_t) : t \in (0, \infty)\}$.*
2. *There is $\xi > \mathbb{R}$ such that*

$$Y = \pi_\Lambda(\mathrm{st}(X_\xi^\sharp + \Lambda^\sharp)).$$

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Summary-Closure vs. Hausdorff limit

The closure of X

We started with $X \subseteq \mathbb{R}^n$ defined over $\mathbb{R}$ and then $\mathrm{cl}(X + \Lambda) = \mathrm{st}(X^\sharp + \Lambda^\sharp)$.

The Hausdorff limits of $\{X_t : t \in (0, \infty)\}$

For each non-standard $\xi > \mathbb{R}$, $\mathrm{st}(X^\sharp_\xi + \Lambda^\sharp)$ is a Hausdorff limit at ∞.

Again, we may partition into types but now not over $\mathbb{R}$, but over $\mathbb{R}\langle\xi\rangle$, the o-minimal structure generated by $\mathbb{R}$ and ξ.
For simplicity, below let $\mathcal{X} = X^\sharp_\xi$.

$$\mathrm{st}(\mathcal{X} + \Lambda^\sharp) = \bigcup_{p \in S_{\mathcal{X}}(\mathbb{R}\langle\xi\rangle)} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$$

Here, $S_{\mathcal{X}}(\mathbb{R}\langle\xi\rangle) =$ the o-minimal types on $\mathcal{X} = X^\sharp_\xi$, over $\mathbb{R}\langle\xi\rangle$.
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Nearest coset to a type

Proposition

For a type $p \in S_n(\mathbb{R}\langle\xi\rangle)$, there is a smallest linear subspace $L_p \subseteq \mathbb{R}^n$, and $\alpha \in \mathbb{R}\langle\xi\rangle$, such that $p(\mathcal{R}) \subseteq \mu + \alpha + L_p^\sharp$.

We call such translate $\alpha + L_p$ a nearest coset of p.

Theorem (Λ-linearity of types)

Assume that $p(x) \in S_n(\mathbb{R}\langle\xi\rangle)$, and $a_p + L_p$ is a nearest coset of p. Then, for every lattice $\Lambda \subseteq \mathbb{R}^n$ we have

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The uniform Hausdorff limits theorem

Theorem

Let $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ be an $\mathbb{R}_{om}$-definable family of subsets of $\mathbb{R}^n$.

Then there are $\mathbb{R}$-linear spaces $L_1, \dots, L_s \subseteq \mathbb{R}^n$, such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

1. If $L_j^\Lambda = \mathbb{R}^n$ for some $j = 1, \dots, s$ then $\mathbb{T}_\Lambda$ is the only Hausdroff limit at ∞ of $\pi_\Lambda(\mathcal{F}) := \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$. This remains true if Λ is replaced by a finite index subgroup.
   Call it $\mathcal{F}$ is strongly Λ-dense in $\mathbb{T}_\Lambda$.
2. If $L_j^\Lambda \neq \mathbb{R}^n$ for all $j = 1, \dots, s$, then exists $K \in \mathbb{N}$ such that for every subgroup $\Lambda_0 \subseteq K \cdot \Lambda$, no Hausdorff limit at ∞ of $\pi_{\Lambda_0}(\mathcal{F})$ is equal to $\mathbb{T}_{\Lambda_0}$.

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The uniform Hausdorff limits theorem

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Let $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ be an $\mathbb{R}_{om}$-definable family of subsets of $\mathbb{R}^n$.

Then there are $\mathbb{R}$-linear spaces $L_1, \ldots, L_s \subseteq \mathbb{R}^n$, such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

1. If $L_j^\Lambda = \mathbb{R}^n$ for some $j = 1, \ldots, s$ then $\mathbb{T}_\Lambda$ is the only Hausdroff limit at ∞ of $\pi_\Lambda(\mathcal{F}) := \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$. This remains true if Λ is replaced by a finite index subgroup.
   Call it $\mathcal{F}$ is strongly Λ-dense in $\mathbb{T}_\Lambda$.
2. If $L_j^\Lambda \neq \mathbb{R}^n$ for all $j = 1, \ldots, s$, then exists $K \in \mathbb{N}$ such that for every subgroup $\Lambda_0 \subseteq K \cdot \Lambda$, no Hausdorff limit at ∞ of $\pi_{\Lambda_0}(\mathcal{F})$ is equal to $\mathbb{T}_{\Lambda_0}$.

Example

$\mathcal{F} : X_t = t + [0, 2] \subseteq \mathbb{R}\,, t \in (0, \infty)$.

Since all X_t are bounded, the only associated nearest coset is $L = \{0\}$, so $L^{\mathbb{Z}} = \{0\} \neq \mathbb{R}$.

Still, for $\Lambda = \mathbb{Z}$¡ or $\Lambda = 2\mathbb{Z}$, for all t, $\pi_\Lambda(X_t) = \mathbb{T}_\Lambda$ (so all Hausdroff limits equal $\mathbb{T}_\Lambda$).

However, if $[\mathbb{Z} : \Lambda] \geqslant 3$ then all the Haudforff limits are partial arcs on the circle $\mathbb{T}_\Lambda$. So, $\mathcal{F}$ is not strongly $\mathbb{Z}$-dense.

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Nilmanifolds

We denote by $U_n(\mathbb{R})$ the group of real upper-triangular $n \times n$ matrices with 1's on the main diagonal.

By a unipotent group we mean a real algebraic subgroup $G \subseteq U_n(\mathbb{R})$. It is a nilpotent group and when abelian $G \cong (\mathbb{R}^k, +)$, for some k.

A lattice in G is a discrete subgroup $\Lambda \subset G$ such that the quotient space G/Λ is compact.

The quotient $M = G/\Lambda$ is called a nilmanifold (it not not a group!).

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The nilmanifold case-reduction to the abelian case

- Let $G_{ab} := G/[G,G]$, an abelian group.
- $\pi_{ab} : G \to G_{ab}$ is the quotient map. For a lattice $\Lambda \subseteq G$, let $\Lambda_{ab} := \pi_{ab}(\Lambda)$ is a lattice in G_{ab}.

Theorem

Let $\mathcal{F} = \{X_t : t \in (0,\infty)\}$ be an $\mathbb{R}_{om}$-definable family of subsets of G. Then for every lattice $\Lambda \subseteq G$, $\mathcal{F}$ is strongly Λ-dense in G/Λ if and only if $\{\pi_{ab}(X_t) : t \in (0,\infty)\}$ is strongly Λ_{ab}-dense in G_{ab}/Λ_{ab}.

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Further problems

- Uniformity of the closure problem, in parameters.
- Equidistribution for $X \subseteq \mathbb{R}^n$ with $\dim X > 1$.
- Describe explicitly all the family of Hausdorff limits of a definable family $\mathcal{F}$ (we only knew then the family is strongly Λ-dense in $\mathbb{T}_\Lambda$).
- Replace unipotent groups by reductive groups: E.g. the closure problem for definable subsets of $SL(n, \mathbb{R})$ and quotients by lattices.