Cardinality and Equivalence Relations

Greg Hjorth

greg.hjorth@gmail.com

University of Melbourne

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1 The broad outline of these talks

1. The basic idea of cardinality. (At the beginning at least, make very few mathematical assumptions of the audience.)

2. Slight modifications of this concept can lead to a spectrum of notions which resemble the notion of *size* or *cardinality*. ("Borel cardinality" or "effective cardinality", or even cardinality in $L(\mathbb{R})$).

3. Some of these notions are implicit in mathematical activity outside set theory. (For instance the work on *dual* of a group by George W. Mackey, or scattered references to *effective cardinality* in the writing of Alain Connes.)

4. In the 80's set theorists such as Harvey Friedman, Alexander Kechris, among others, began to suggest a way to explicate this idea around the concept of *Borel reducibility*

5. The outpouring of activity in this area over the last 15 -20 years. Su Gao, Alexander Kechris, Alain Louveau, Slawek Solecki, Simon Thomas, Boban Velickovic, among many others.

6. Dichotomy theorems for Borel equivalence relations (for instance, the Harrington, Kechris, and Louveau extension of Glimm-Effros).

7. Specific classification problems in mathematics (for instance the finite rank torsion free abelian groups)

8. Dynamical methods to analyze equivalence relations in the absence of reasonable dichotomy theorems (for instance turbulence)

9. Interactions between the theory of countable equivalence relations and the theory of orbit equivalence

2 Equivalence relations and invariants

Definition Let X be a set. $E \subset X \times X$ is said to be an *equivalence relation* if

1. it is reflexive (xEx all $x \in X$)
2. symmetric (xEy implies yEx)
3. transitive (xEy along with yEz implies xEz).

Example 1. Let X be the set of people. Let E be the equivalence relation of having the same height.

2. Let X again be the set of all people. Let E be the equivalence relation of having the same mother.

3. Let X be the set of all planets. Let E be the equivalence relation of being located in the same universe.

Definition For E an equivalence relation on a set X, a *complete invariant* or *classification* of E is a "reasonable" or "explicit" or "natural" function

$$f : X \to I$$

such that for all $x_1, x_2 \in X$ we have

$$x_1 E x_2$$

if and only if

$$f(x_1) = f(x_2).$$

This really a pseudo-definition, held completely hostage to how we best make sense of "reasonable" or "explicit".

Part of the story are the attempts to make this idea more precise, and this in turn connects in with variations on the concept of *cardinality*

Example 1. For E the equivalence relation of having the same height we clearly do have a natural classification. Namely: Height measured in feet and inches. Assign to each x in the set of people the height measured in feet and inches as $f(x)$.

2. For E the equivalence relation of having the same mother, it would likewise seem that there is a very explicit invariant: Assign to each person x their mother as $f(x)$.

3. For E being the equivalence relation of being in the same universe, the situation is not *so* clear.

   We could try to for instance assign to each planet the actual universe in which they live, but it is not clear that this is doing much more than assigning to each x the entire set of all y for which xEy.

Definition A function

$$f : X \to Y$$

is an *injection* if for all $x_1, x_2 \in X$,

$$x_1 \neq x_2$$

implies

$$f(x_1) \neq f(x_2).$$

In other words, an injection is a function which sends distinct points to distinct images.

Example 1. Let X be the set of people and let Y be the set of all human heads. Let

$$f : X \to Y$$

assign to each x its head. This is presumably (barring some very unusual case of siamese twins) an injection.

2. Let X be the set of all people who have ever lived and let

$$f : X \to X$$

assign to each X its mother. (Here we are ignoring minor chicken-egg questions about the first ever mother). This is clearly not an injection. There are cases of distinct people having the same mother.

Definition A function

$$f : X \to Y$$

is a *surjection* if for each $y \in Y$ there is some $x \in X$ with

$$f(x) = y.$$

A function which is both injective and surjective is called a *bijection*.

In rough terms, a bijection between X and Y is a way of marrying all the X's off with all the Y's with no unmarried Y's left over. (Here assuming no polygamy allowed).

Definition Given two sets X and Y, we say that the *cardinality of* X *is less than the cardinality of* Y, written

$$|X| \leq |Y|,$$

if there is an injection

$$f : X \to Y.$$

Theorem 3.1 *(Schroeder-Bernstein) If*

$$|X| \leq |Y|$$

and

$$|Y| \leq |X|,$$

then there is a bijection between the two sets.

Intuitively not so outrageous. If we say that a set has size 4, and count of the elements 1, 2, 3, 4, we are implicitly placing that set in a bijection with the set $\{1, 2, 3, 4\}$.

Definition A set α is an *ordinal* if:

1. it is transitive – $\beta \in \alpha$ along with $\gamma \in \beta$ implies $\gamma \in \alpha$; and
2. it is linearly ordered by $\in$ – if β, γ are both in α, then
$$\beta \in \gamma,$$
or
$$\gamma \in \beta,$$
or
$$\gamma = \beta.$$

$\emptyset$, the set having no members, which set theorists customarily identify with 0.

$1 = \{0\}$, the set whose only member is 0.

$2 = \{0, 1\}$, gives the usual set theoretical definition of 2. Then we keep going with $3 = \{0, 1, 2\}$, $4 = \{0, 1, 2, 3\}$, and so on.

We reach the first infinite ordinal with the set of natural numbers:

$$\omega = \{0, 1, 2, ...\}.$$

This again leads to a whole new ladder of ordinals:

$$\omega + 1 = \{0, 1, 2, ..., \omega\},$$
$$\omega + 2 = \{0, 1, 2, ..., \omega, \omega + 1\},$$
$$\omega + 3 = \{0, 1, 2, ..., \omega, \omega + 1, \omega + 2\},$$

and onwards:

$$\omega + \omega = \{0, 1, 2, ..., \omega, \omega + 1, \omega + 2, ...\},$$
$$\omega+\omega+1 = \{0, 1, 2, ..., \omega, \omega+1, \omega+2, ..., \omega+\omega\},$$
$$\omega\times\omega = \{0, 1, 2, ..., \omega, \omega+1, ...\omega+\omega, ...\omega+\omega+\omega, ...\},$$

ad infinitum.

Lemma 4.1 *The ordinals themselves are linearly ordered: If β, γ are both ordinals, then*

$$\beta \in \gamma,$$

or

$$\gamma \in \beta,$$

or

$$\gamma = \beta.$$

The Axiom of Choice: (In effect) Every set can be placed in a bijection with some ordinal.

Definition An ordinal is said to be a *cardinal* if it cannot be placed in a bijection with any smaller ordinal.

So for instance, ω (or $\aleph_0$ as it is sometimes called in this context) is indeed a cardinal: The smaller ordinals are finite.

But not $\omega + \omega$: Define

$$f : \omega + \omega \longrightarrow \omega,$$

$$n \mapsto 2n,$$

$$\omega + n \mapsto 2n + 1.$$

The consequences of all this for the theory of cardinality:

> Every set can be placed in a bijection with an ordinal.
>
> The cardinals are a linearly ordered set.

This is parallel to a form of utilitarianism: There is only one good (human happiness) and that good can be compared in order and amount.

There is only one notion of "size" and the cardinals can be compared in order.

Recall that we set $|X| \leq |Y|$ if there is an injection from X to Y.

However, it does make sense to look at parallel definitions for classes of injections which are more narrow than simply the class of *all* injections.

Fix Γ some class of functions (for instance, all Borel functions, all functions in $L(\mathbb{R})$).

We might want to say that the Γ-*cardinality of X is less than equal to that of Y* if there is some injection $f \in \Gamma$ with

$$f : X \to Y.$$

This should be compared to the problem of classifying an equivalence relation.

Definition For E and equivalence relation on X, and $x \in X$, let $[x]_E$ be the set of all $y \in X$ with

$$xEy.$$

Then let X/E be the set of all equivalence classes:

$$\{[x]_E : x \in X\}.$$

For instance, if X is the set of all people, and E is the equivalence relation of having the same height, then

$$[\text{Greg}]_E$$

would be the set of all people who are 5'6" tall.

X/E would consist of the set of all "groupings" of people where they are categorized strictly by height.

Then a classification of E would in some form be an "appropriate" function

$$f : X \to I$$

which induces an injection

$$\hat{f} : X/E \to I$$

via letting

$$\hat{f}(C)$$

take the value

$$f(y)$$

for any y in the equivalence class C.

The assumption $xEy \Rightarrow f(x) = f(y)$ ensures $\hat{f}$ is well defined.

The assumption $f(x) = f(y) \Rightarrow xEy$ ensures $\hat{f}$ is an injection.

However it remains to resolve the definition of what we should count as an "appropriate" or reasonable class of possible functions f.

One particular approach to theory of what should count as *reasonable* functions for the point of view of classification has been suggested by work of Mackey on group representations dating back to the middle of the last century.

The story now becomes considerably more mathematical. I will start first by describing the problem Mackey considered, and only then the explication of *reasonable* his work suggests.

Definition For H a Hilbert space, $U(H)$ denotes the group of *unitary operators* on H. That is to say, the set of all linear bijections

$$T : H \to H$$

such that for all $u, v \in H$

$$\langle T(v), T(u) \rangle = \langle u, v \rangle.$$

Definition For G a countable group, a *unitary representation of* G (on the Hilbert space H) is a homomorphism

$$\varphi : G \to U(H)$$

$$g \mapsto \varphi_g.$$

We then say that a representation is *irreducible* if the only closed subspaces of H which are invariant under

$$\{\varphi_g : g \in G\}$$

are the trivial ones: 0 and H.

Let $\mathrm{Irr}(G, H)$ denote the collection of irreducible unitary representations of G on H.

Definition Two unitary representations

$$\varphi : G \to U(H_\varphi)$$
$$\psi : G \to U(H_\psi)$$

are *equivalent*, written

$$\varphi \cong \psi,$$

if they are *unitarily conjugate* in the sense that there is a unitary isomorphism

$$T : H_\varphi \to H_\psi$$

such that at every $g \in G$

$$\varphi_g = T \circ \psi_g \circ T^{-1}.$$

Theorem 6.1 *If $\varphi : \mathbb{Z} \to U(H)$ is an irreducible representation, then:*

1. *H is one dimensional; and*
2. *there is some $z \in \mathbb{C}$ such that at every $\ell \in \mathbb{Z}$ we have*
$$\varphi_\ell(v) = z^\ell \cdot v$$
all $v \in H$; and
3. *two distinct irreducible representations are equivalent if and only if they have the same $z \in \mathbb{C}$ associated to them.*

Thus we obtain a complete classification of irreducible representations of $\mathbb{Z}$ by their associated $z \in \mathbb{C}$. It is like we can view $\mathrm{Irr}(\mathbb{Z}, H)/\cong$ as a subset of $\mathbb{C}$.

It turns out that a similar, though somewhat more complicated, classification can be given for any abelian group.

In broad terms Mackey was led to ask: For which groups G can we *reasonably classify* the collection of equivalence classes

$$\mathrm{Irr(G, H)} / \cong$$

by points in some *concrete* space such as $\mathbb{C}$?

Before even groping towards an answer, one might first want to make the question precise.

Mackey did make the question precise, but this in turn requires the introduction of ideas lying at the foundations of descriptive set theory.

Definition A topological space is said to be *Polish* if it is separable and it admits a complete compatible metric.

We then say that the *Borel sets* are those appearing in the smallest σ-algebra containing the open sets.

A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology giving rise to that σ-algebra as its collection of Borel sets.

A function between two Polish spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ the pullback $f^{-1}[B]$ is Borel.

Some examples

1. Any separable Hilbert space is Polish.
2. If H is a separable Hilbert space, then $U(H)$ is a closed subgroup of its isometry group and hence Polish.
3. If H is a separable Hilbert space and G is a countable group, then

$$\prod_G U(H)$$

is a countable product of Polish spaces and hence Polish.

4. Then the collection of unitary representations of G is a closed subspace of $\prod_G U(H)$, and hence Polish.
5. Finally it is a slightly non-trivial fact that the collection of irreducible representations is a G_δ subset of the collection of all representations, and hence Polish.

Definition (Mackey) An equivalence relation E on a Polish space X is *smooth* if there is a another Polish space Y and a Borel function

$$f : X \to Y$$

such that for all $x_1, x_2 \in X$ we have

$$f(x_1) = f(x_2)$$

if and only if

$$x_1 E x_2.$$

A countable group G has *smooth dual* if for any separable Hilbert space H, the equivalence relation $\cong$ on $\mathrm{Irr}(G, H)$ is smooth.

Question (Mackey, in effect) Which groups have smooth dual?

There are various answers in the literature to Mackey's question, including Glimm's solution of the *Mackey conjecture*, which applies not just to discrete groups but more generally lcsc topological groups.

In the case of discrete groups there is a completely algebraic characterization.

Theorem 7.1 *(Thoma) A countable group G has smooth dual if and only if it has an abelian subgroup with finite index.*

There are some very simple, from the point of view of Borel complexity, non-smooth equivalence relations.

Definition Equip

$$2^{\mathbb{N}} =_{\mathrm{df}} \prod_{\mathbb{N}} \{0, 1\}$$

with the product topology.

Let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$.

Lemma 7.2 E_0 *is not smooth.*

E_0 itself is F_σ as a subset of $2^{\mathbb{N}}$. The complexity of its classification problem has little do with any complexity it might have as a subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

It turns out that at the base of Glimm's proof of the Mackey conjecture is a theorem to the effect that under certain circumstances E_0 is the *canonical* obstruction to smoothness.

This was generalized by Ed Effros.

The final and ultimate generalization to the abstract theory of Borel equivalence relations was obtained by Leo Harrington, Alexander Kechris, and Alain Louveau in the late 1980's and in turn sparked a new direction of research in descriptive set theory.

Classification Problems in Mathematics

Greg Hjorth

greg.hjorth@gmail.com

University of Melbourne

Tarski Lectures

University of California, Berkeley

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1 Recall

Definition The *cardinality of X is less than or equal to Y*,

$$|X| \leq |Y|,$$

if there is an injection from X to Y.

This might suggest a notion of “cardinality” where we restrict our attention to some restricted class of injections.

This in turn could relate to the idea that an equivalence relation E on a set X is in some sense *classifiable* if there is a “reasonably nice” or “natural” or “explicit” function

$$f : X \to I$$

which induces (via $x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2)$) an injection

$$\hat{f} : X/E \to I.$$

In the context of unitary group representation a definition exactly along these lines was proposed by G. W. Mackey.

Definition (Mackey) An equivalence relation E on a Polish space X is *smooth* if there is a Polish space Y and a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow f(x_1) = f(x_2).$$

In the way of context and background

1. Borel functions are considered by many mathematicians to be basic and uncontroversial, and concrete in a way that a function summoned in to existence by appeal to the axiom of choice would not.

2. Many classification problems can be cast in the form of understanding an equivalence relation on a Polish space

In the late 80's two pivotal papers suggested a variation and generalization of Mackey's definition.

A Borel Reducibility Theory for Classes of Countable Structures, H. Friedman and L. Stanley, **The Journal of Symbolic Logic**, Vol. 54, No. 3 (Sep., 1989), pp. 894-914

A Glimm-Effros Dichotomy for Borel Equivalence Relations, L. A. Harrington, A. S. Kechris and A. Louveau, **Journal of the American Mathematical Society**, Vol. 3, No. 4 (Oct., 1990), pp. 903-928

Neither paper referenced the other, and yet they used the exact same terminology and notation to introduce a new concept.

Definition Given equivalence relations E and F on X and Y we say that E is Borel reducible to F, written

$$E \leq_B F,$$

if there is a Borel function

$$f : X \to Y$$

such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

In other words, the Borel function f induces an injection

$$\hat{f} : X/E \to Y/F.$$

The perspective of Friedman and Stanley was to compare various classes of countable structures under the ordering $\leq_B$. The Harrington, Kechris, Louveau paper instead generalized earlier work of Glimm and Effros in foundational issues involving the theory of unitary group representations.

Definition Let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$. For X a Polish space let $\mathrm{id}(X)$ be the equivalence relation of equality on X.

Thus in the above notation we can recast Mackey's definition of *smooth*: An equivalence relation E is smooth if for some Polish X we have

$$E \leq_B \mathrm{id}(X).$$

It turns out that for any uncountable Polish space X we have

$$\mathrm{id}(\mathbb{R}) \leq_B \mathrm{id}(X)$$

and

$$\mathrm{id}(X) \leq_B \mathrm{id}(\mathbb{R}).$$

Definition An equivalence relation E on Polish X is *Borel* if it is Borel as a subset of $X \times X$.

Theorem 2.1 *(Harrington, Kechris, Louveau) Let E be a Borel equivalence relation. Then exactly one of the following two conditions holds:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$*;*

2. $E_0 \leq_B E$*.*

Moreover 1. is equivalent to E being smooth.

This breakthrough result, this archetypal *dichotomy theorem*, suggested the possibility of understanding the structure of the Borel equivalence relations up to Borel reducibility, which in turn has become a major project in the last twenty years, which I will survey on Friday.

3 Examples of Borel reducibility in mathematical practice

It turns out that many classical, or near classical, theorems can be recast in the language of Borel reducibility.

Example Let (X, d) be a complete, separable, metric space. Let $K(X)$ be the compact subsets of X – equipped with the metric $D(K_1, K_2)$ equals

$$\sup_{x \in K_1} d(x, K_2) + \sup_{x \in K_2} d(x, K_1),$$

where $d(x, K) = \inf_{z \in K} d(x, z)$.

Let E be the equivalence relation of isometry on $K(X)$. Then Gromov showed that E is smooth. In other words,

$$E \leq_B \mathrm{id}(\mathbb{R}).$$

Example Let H be a separable Hilbert space and $U(H)$ the group of unitary operators of H.

Let $\cong$ be the equivalence relation of conjugacy on $U(H)$, which is in effect the isomorphism relation considered in the last talk: $T_1 \cong T_2$ if

$$\exists S \in U(H)(S \circ T_1 \circ S^{-1} = T_2).$$

1. In the case that H is finite dimensional, every $T \in U(H)$ can be diagonalized. This gives a reduction of $\cong$ to the equality of finite subsets of $\mathbb{C}$, and hence a proof that $\cong$ is smooth.

2. In the case that H is infinite dimensional, the situation is considerably more subtle, but the spectral theorem allows us to write each element of $U(H)$ as a kind of direct integral of rotations.

Definition Let S^1 be the circle:

$$\{z \in \mathbb{C} : |z| = 1\}$$

in the obvious, and compact, topology. Let $P(S^1)$ be the collection of probability measures on S^1 – this forms a Polish space in the topology it inherits from being a closed subset $C(S^1)^*$ in the weak star topology (via the Riesz representation theorem). For $\mu, \nu \in P(S^1)$, set $\mu \sim \nu$ if they have the same null sets.

It then follows from the spectral theorem that

$$\cong \leq_B \sim .$$

The spectral theorem is often considered, though without the use of the language of Borel reducibility, to provide a *classification* of the infinite dimensional unitary operators up to conjugacy.

Example For S a countable set, may identify $\mathcal{P}(S)$ with

$$2^S = \prod_S \{0,1\}$$

and thus view it as a compact Polish space in the product topology.

A torsion free abelian (TFA) group A is said to be of rank $\leq n$ if there are $a_1, a_2, ...a_n \in A$ such that every $b \in A$ has some $m \in \mathbb{N}$ with $m \cdot b \in \langle a_1, ..., a_n \rangle$.

Up to isomorphism, the rank $\leq n$ TFA groups are exactly the subgroups of $(\mathbb{Q}^n, +)$, and thus form a Polish space as a subset of $\mathcal{P}(\mathbb{Q}^n)$.

Let $\cong_n$ be the isomorphism relation on subgroups of $(\mathbb{Q}^n, +)$. In the language of Borel reducibility a celebrated classification theorem can be rephrased as:

Theorem 3.1 *(Baer)* $\cong_1 \leq_B E_0$.

Example Let $\mathrm{Hom}^+([0,1])$ be the orientation preserving homeomorphisms of the closed unit interval. In the sup norm metric, this forms a Polish space.

Let $\cong_{\mathrm{Hom}^+([0,1])}$ be the equivalence relation of conjugacy.

There is a kind of folklore observation to the effect that every element of $\mathrm{Hom}^+([0,1])$ can be classified *symbolically*, by recording the maximal open intervals on which it is increasing, decreasing, or the identity.

This translates into classifying

$$\mathrm{Hom}^+([0,1])/\cong_{\mathrm{Hom}^+([0,1])}$$

by countable linear orderings with equipped with unary predicates P_{inc} and P_{dec} up to isomorphism. Those in turn can be viewed as forming a closed subset of $2^{\mathbb{N}\times\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, and we obtain

$$\cong_{\mathrm{Hom}^+([0,1])} \leq_B \cong_{2^{\mathbb{N}\times\mathbb{N}}\times 2^{\mathbb{N}}\times 2^{\mathbb{N}}}.$$

Definition Let $\mathcal{L}$ be a countable language. Then $\mathrm{Mod}(\mathcal{L})$ is the set of all $\mathcal{L}$ structures with underlying set $\mathbb{N}$.

Definition Let τ_{qf} be the topology with basic open sets of the form

$$\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{M} \models \varphi(\vec{a})\}$$

where $\varphi(\vec{x})$ is quantifier free and $\vec{a} \in \mathbb{N}^{<\infty}$.

τ_{fo} is defined in a parallel fashion, except with $\varphi(\vec{x})$ ranging over first order formulas, and more generally for $F \subset \mathcal{L}_{\omega_1,\omega}$ a countable *fragment* we define τ_F similarly with $\varphi(\vec{x}) \in F$.

It is not much more than processing the definitions to show τ_{qf} is Polish. For instance for $\mathcal{L}$ consisting of a single binary relation, we obtain a natural isomorphism with $2^{\mathbb{N}\times\mathbb{N}}$. It can be shown, however, that the others are Polish, and all these examples have the same Borel structure.

Definition For a sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ we let $\cong_\sigma$ be isomorphism on $\mathrm{Mod}(\sigma)$, the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ with $\mathcal{M} \models \sigma$.

$\cong_\sigma$ is *universal for countable structures* if given any countable language $\mathcal{L}'$ we have

$$\cong_{\mathrm{Mod}(\mathcal{L}')} \leq_B \cong_\sigma .$$

Theorem 4.1 *(Friedman, Stanley) The following are universal for countable structures: Isomorphism of countable trees, countable fields, and countable linear orderings. Isomorphism of countable* torsion *abelian groups is not universal for countable structures.*

One tends to obtain universality[1] for such a class of countable structures , except when there is an "obvious" reason why this must fail.

For instance, if the isomorphism relation is "essentially countable".

[1]The major being torsion abelian groups. The case for torsion free abelian groups remains puzzlingly open, despite strong indicators it should be universal

Definition A Borel equivalence relation F on Polish Y is *countable* if every equivalence class is countable.

An equivalence relation E on a Polish space is *essentially countable* if it is Borel reducible to a countable equivalence relation.

An equivalence relation E is *universal for essentially countable* if it is essentially countable and for any other countable Borel equivalence F we have $F \leq_B E$.

Theorem 5.1 *(Jackson, Kechris, Louveau) Universal essentially countable equivalence relations exist. In fact, for* $\mathbb{F}_2$ *the free group on two generators, the orbit equivalence relation of* $\mathbb{F}_2$ *on* $2^{\mathbb{F}_2}$ *is essentially countable.*

Fact 5.2 *If an equivalence relation E is essentially countable, then for some*[2] *countable languages $\mathcal{L}$ we have $E \leq_B \cong_{\mathrm{Mod}(\mathcal{L})}$.*

Theorem 5.3 *(Kechris) If G is a locally compact Polish group acting in a Borel manner on a Polish space X, then the resulting equivalence relation is essentially countable.*

In the context of isomorphism types of classes of countable structures, one can characterize when an equivalence relation is essentially countable in model theoretic terms.

Roughly speaking a class of countable structures with an appropriately "finite character" will be essentially countable.

In particular, if $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ satisfying σ is finitely generated, then $\cong_\sigma$ is essentially countable.

[2] In fact, "most"

Theorem 5.4 *(Thomas, Velickovic) Isomorphism of finitely generated groups is universal for essentially countable.*

As in the case of general countable structures, the tendency is for classes of essentially countable structures to be universal unless there is some relatively obvious obstruction.

In a paper with Kechris, we made a rather arrogant, reckless, and totally unsubstantiated, conjecture that isomorphism for rank two torsion free abelian groups would be universal for essentially countable.

Since E_0 is *not* universal for essentially countable, this was hoped to explain the inability of abelian group theorists to find a satisfactory classification for the higher finite rank torsion free abelian groups.

6 The saga of finite rank torsion free abelian groups

Although many well known mathematical classification theorems have a direct consequence for the theory of Borel reducibility, a major motivation has been to use the theory of Borel reducibility to explicate basic obstructions to the classification of certain classes of isomorphism.

One of the most clear cut cases has been the situation with finite rank torsion free abelian groups.

Recall that $\cong_n$ is being used to describe the isomorphism relation on rank $\leq n$ torsion free abelian groups, where we provide a model of the full set of isomorphism types by considering the subgroups of $\mathbb{Q}^n$.

Theorem 6.1 *(Baer, implicitly, 1937)*

$$\cong_1 \leq_B E_0.$$

A kind of mathematically precise justification for the vague feeling that rank two torsion free abelian groups did not admit a similar classification was provided by:

Theorem 6.2 *(Hjorth, 1998)* $\cong_2$ *is* not *Borel reducible to* E_0*.*

In some sense this addressed the soft philosophical motivation behind the conjecture with Kechris, but not the hard mathematical formulation with which it faced the world. This was left to Simon Thomas, who in a technically brilliant sequence of papers showed:

Theorem 6.3 *(Thomas, 2002, 2004) At every* n

$$\cong_n <_B \cong_{n+1} .$$

In general, and this lies at the heart of the technical mountains Thomas had to overcome, almost all the results to show that one essentially countable equivalence relation is *not* Borel reducible to another rely on techniques coming entirely outside logic, such as geometric group theory, von Neumann algebras, and the rigidity theory one finds in the work of Margulis and Zimmer.

In recent years the work of logicians in this area has begun to communicate and interact with mathematicians in quite diverse fields.

However, it has gradually become clear that many of the problems we would most dearly like to solve will not be solvable by the measure theoretic based techniques being used in these other fields. For instance....

Question Let G be a countable nilpotent group acting in a Borel manner on a Polish space with induced orbit equivalence relation E_G. Must we have

$$E_G \leq_B E_0?$$

The problem here is that with respect to any measure we will have $E_G \leq_B E_0$ on some conull set, and thus measure will not be a suitable method for proving the existence of a counterexample. In an enormously challenging and strikingly original fifty page manuscript, Su Gao and Steve Jackson showed $E_G \leq_B E_0$ when G is abelian.

Many of these issues relate to open problems in the theory of Borel dichotomy theorems and the global structure of the Borel equivalence relations under $\leq_B$.[3]

[3] Next talk

Definition An equivalence relation E on a Polish space X is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$f : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) \cong f(x_2).$$

Here one might compare algebraic topology, where algebraic objects considered up to isomorphism are assigned as invariants for classes of topological spaces considered up to homeomorphism.

Again it turns out that some well known classification theorems have the direct consequence of showing that some naturally occurring equivalence relation admits classification by countable structures.

Example Recall $\cong_{\mathrm{Hom}^+([0,1])}$ as the isomorphism relation on orientation measure preserving transformations of the closed unit interval. Then the folklore observation mentioned from before in particular shows that $\cong_{\mathrm{Hom}^+([0,1])}$ is classifiable by countable structures.

Example A *Stone space* is a compact zero dimensional Hausdorff space. There is a fixed topological space X (for instance, the Hilbert cube), such that every separable Stone space can be realized as a compact subspace of X. Then $S(X)$, the set of all such subspaces, forms a standard Borel space, and we can let $\cong_{S(X)}$ be the homeomorphism relation on elements of $S(X)$.

Stone duality, the classification of Stone spaces by their associated Boolean algebras, in particular shows that $\cong_{S(X)}$ is classifiable by countable structures.

Example For λ the Lebesgue measure on $[0, 1]$, let M_∞ be the group of measure preserving transformations of $([0, 1]\lambda)$ (considered up to equality a.e.).

A measure preserving transformation T is said to be *discrete spectrum* if $L^2([0, 1], \lambda)$ is spanned by eigenvalues for the induced unitary operator

$$U_T : f \mapsto f \circ T^{-1}.$$

It follows from the work of Halmos and von Neumann that such transformations considered up to conjugacy in M_∞ are classifiable by countable structures.

Example The search for complete algebraic invariants has recurrent theme in the study of C^*-algebras and topological dynamics.

Consider minimal (no non-trivial closed invariant sets) homeomorphisms of $2^{\mathbb{N}}$

Let $\sim_{C(2^{\mathbb{N}})}$ be conjugacy of orbit equivalence relations: Thus

$$f_1 \sim_{C(2^{\mathbb{N}})} f_2$$

if there is some homeomorphism g conjugating their orbits:

$$\forall \vec{x}(g[\{f_1^{\ell}(\vec{x}) : \ell \in \mathbb{Z}\}] = \{f_2^{\ell}(g(\vec{x})) : \ell \in \mathbb{Z}\}).$$

Giordano, Putnam, and Skau produce countable ordered abelian groups which, considered up to isomorphism, act as complete invariants.

Their theorem implicitly shows $\sim_{C(2^{\mathbb{N}})}$ to be classifiable by countable structures.

This a theory, or rather a body of techniques, explicitly fashioned to show when equivalence relations are *not* classifiable by countable structures.

Definition Let G be a Polish group acting continuously on a Polish space X. For V an open neighborhood of 1_G, U an open set containing x, we let

$$O(x, U, V),$$

the U-V-local orbit, be the set of all $\hat{x} \in [x]_G$ such that there is a finite sequence

$$(x_i)_{i \leq k} \subset U$$

such that

$$x_0 = x, \qquad x_k = \hat{x},$$

and each

$$x_{i+1} \in V \cdot x_i.$$

Definition Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

1. every orbit is dense; and
2. every orbit is meager; and
3. for $x \in X$, the local orbits of x are all somewhere dense; that is to say, if V is an open neighborhood of 1_G, U is an open set containing x, then closure of $O(x, U, V)$ contains an open set.

Theorem 8.1 *(Hjorth) Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G.*

If G acts turbulently on X, then E_G is not *classifiable by countable structures.*

This has been the engine behind a number of anti-classification theorems.

Example (Kechris, Sofronidis) Infinite dimensional unitary operators considered up to unitary conjugacy do not admit classification by countable structures.

Example (Hjorth) The homeomorphism group of the unit square,

$$\mathrm{Hom}([0,1]^2),$$

considered up to homeomorphism does not admit classification by countable structures.

Example (Gao) Countable metric spaces up to homeomorphism does not admit classification by countable structures.

Example (Törnquist) Measure preserving actions of $\mathbb{F}_2$ up to orbit equivalence do not admit classification by countable structures.

Borel Equivalence Relations: Dichotomy Theorems and Structure

Greg Hjorth

greg.hjorth@gmail.com

University of Melbourne

Tarski Lectures

University of California, Berkeley

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Definition A space is *Polish* if it is separable and admits a complete metric.

We then say that the *Borel sets* are those appearing in the smallest σ-algebra containing the open sets.

A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology giving rise to that σ-algebra as its collection of Borel sets.

A function between two Polish spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ the pullback $f^{-1}[B]$ is Borel.

We have gone through a number of examples in the first two talks. There is a sense in which Polish spaces are ubiquitous.

The notion of standard Borel space is slightly more subtle.

However it turns out that there are many examples of standard Borel spaces which possess a *canonical* Borel structure, but no *canonical* Polish topology.[1]

Theorem 1.1 *(Classical) If X is a Polish space and $B \subset X$ is a Borel set, then B (equipped in the σ-algebra of Borel subsets from the point of view of X) is standard Borel.*

Theorem 1.2 *(Classical; the "perfect set theorem") If X is a Polish space and $B \subset X$ is a Borel set, then exactly one of:*

1. *B is countable; or*
2. *B contains a homeomorphic copy of Cantor space, $2^{\mathbb{N}}$ (and hence has size $2^{\aleph_0}$).*

[1]Indeed, since we are mostly only considering Polish spaces up to questions of Borel structure, it is natural to discount the specifics of the Polish topology involved.

Theorem 1.3 *(Classical) If X is a standard Borel space, then the cardinality of X is one of* $\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}$.

Moreover!

Theorem 1.4 *(Classical) Any two standard Borel spaces of the same cardinality are Borel isomorphic.*

Here we say that X and Y are Borel isomorphic if there is a Borel bijection

$$f : X \to Y$$

whose inverse is Borel.[2]

Thus, as sets equipped with their σ-algebras they are isomorphic.

There is a similar theorem for quotients of the form X/E, E a Borel equivalence relation.

[2]In fact it is a classical theorem that any Borel bijection must have a Borel inverse

2 The analogues for Borel equivalence relations

Definition If X is a standard Borel space, an equivalence relation E on X is *Borel* if it appears in the σ-algebra on $X \times X$ generated by the rectangles $A \times B$ for A and B Borel subsets of X.

Theorem 2.1 *(Silver, 1980) Let X is a standard Borel space and assume E is a Borel equivalence relation on X. Then the cardinality of X/E is one of*

$$\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}.$$

However here there is no *moreover.*

In terms of Borel structure, and the situation when X/E is uncountable, there are vastly many possibilities at the level of Borel structure.

Definition Given equivalence relations E and F on standard Borel X and Y we say that E is Borel reducible to F, written

$$E \leq_B F,$$

if there is a Borel function

$$f : X \to Y$$

such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

We say that the *Borel cardinality* X/E *is less than the Borel cardinality of* Y/F, written

$$E <_B F,$$

if there is a Borel reduction of E to F but no Borel reduction of F to E.

In the language Borel reducibility, there is a sharper version of Silver's theorem, which he also proved without describing it in these terms.

Theorem 2.2 *(Silver) Let E be a Borel equivalence relation on a standard Borel space. Then exactly one of:*

1. $E \leq_B \mathrm{id}(\mathbb{N})$*; or*

2. $\mathrm{id}(\mathbb{R}) \leq_B E$*.*

One of the major events in the prehistory of the subject is Leo Harrington's alternate and far shorter proof of Silver's result using a technology called *Gandy-Harrington forcing*.

Building on this technology with the combinatorics of an earlier argument due to Ed Effros, the whole field of Borel equivalence relations was framed by the landmark theorem of Harrington, Kechris, and Louveau.

Recall that E_0 is the equivalence relation of eventual agreement on infinite binary sequences.

Theorem 2.3 *(Harrington, Kechris, Louveau, 1990) Let E be a Borel equivalence relation on a standard Borel space. Then exactly one of:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$*; or*

2. $E_0 \leq_B E$.

This raised the fledgling hope that we might be able to provide a kind of structure theorem for the Borel equivalence relations under $\leq_B$, but before recounting this part of the tale I wish to describe the analogies which exist in the theory of $L(\mathbb{R})$ cardinality.

3 Cardinality in $L(\mathbb{R})$

Definition $L(\mathbb{R})$ is the smallest model of ZF containing the reals and the ordinals.

Although this quick formulation finesses out of the need to provide any set theoretical formalities, it rather disguises the true nature of this inner model.

It turns out that $L(\mathbb{R})$ can be defined by simply closing the reals under certain kinds of highly "constructive" operations carried out through transfinite length along the ordinals. It should possibly be thought of as the collection of sets which can be defined "internally" or "primitively" from the reals and the ordinals.

In this talk I want to think of it as a class inner model which contains anything one might think of as being a necessary consequence of the existence of the reals.

It also turns out that ZFC is incapable of deciding even the most basic questions about the theory and structure of $L(\mathbb{R})$.

On the other hand, if $L(\mathbb{R})$ satisfies AD, or the "Axiom of Determinancy" then almost all those ambiguities are resolved.

Following work of work of Martin, Steel, Woodin, and others, we now know that any reasonably large "large cardinal assumption" implies $L(\mathbb{R}) \models$ AD.

This along with the fact that $L(\mathbb{R}) \models$ AD has many regularity properties displayed by the Borel sets (such as the perfect set theorem for arbitrary sets in standard Borel spaces, all sets of reals Lebesgue measurable) has convinced many set theorists, though not all, that this is the right assumption under which to explore its structure.

I am *not* going to ask the audience to necessarily accept this perspective. I am simply going to examine the cardinality theory of $L(\mathbb{R})$ as a kind of idealization of the theory of Borel cardinality.

From now on in this part I will assume

$$L(\mathbb{R}) \models \mathrm{AD}.$$

Definition For A and B in $L(\mathbb{R})$, we say that the $L(\mathbb{R})$ *cardinality of* A *is less than or equal to the* $L(\mathbb{R})$ *cardinality of* B, written

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})},$$

if there is an injection in $L(\mathbb{R})$ from A to B. Similarly

$$|A|_{L(\mathbb{R})} < |B|_{L(\mathbb{R})},$$

if there is an injection in $L(\mathbb{R})$ from A to B but not from B to A.

Since the axiom of choice fails inside $L(\mathbb{R})$, there is no reason to imagine that the $L(\mathbb{R})$ cardinals will be linearly ordered, and in fact there *are* incomparable cardinals inside $L(\mathbb{R})$.

It turns out that the theory of $L(\mathbb{R})$ cardinality simulates and extends the theory of Borel cardinality.

In every significant case, the proof that

$$E <_B F$$

has also given a proof that

$$|X/E|_{L(\mathbb{R})} < |Y/F|_{L(\mathbb{R})}.$$

This is partially explained by:

Fact 3.1 *For E and F Borel equivalence relations one has*

$$E \leq_{L(\mathbb{R})} F$$

if and only if

$$|\mathbb{R}/E|_{L(\mathbb{R})} \leq |\mathbb{R}/F|_{L(\mathbb{R})}.$$

The two dichotomy theorems for Borel equivalence relations allow a kind of extension to the cardinality theory of $L(\mathbb{R})$.

Theorem 3.2 *(Woodin) Let* $A \in L(\mathbb{R})$. *Then exactly one of the following two things must happen:*

1. $|A|_{L(\mathbb{R})} \leq |\alpha|_{L(\mathbb{R})}$, *some ordinal* α*; or*

2. $|\mathbb{R}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

Theorem 3.3 *(Hjorth) Let* $A \in L(\mathbb{R})$. *Then exactly one of the following two things must happen:*

1. $|A|_{L(\mathbb{R})} \leq |\mathcal{P}(\alpha)|_{L(\mathbb{R})}$, *some ordinal* α*; or*

2. $|\mathcal{P}(\omega)/\mathrm{Fin}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

Here $|\mathcal{P}(\omega)/\mathrm{Fin}|_{L(\mathbb{R})} = |2^{\mathbb{N}}/E_0|_{L(\mathbb{R})}$, thus providing an analogy with Harrington-Kechris-Louveau.

4 Further structure

Definition Let E_1 be the equivalence relation of eventual agreement on $\mathbb{R}^{\mathbb{N}}$. For $\vec{x}, \vec{y} \in (2^{\mathbb{N}})^{\mathbb{N}}$, set $\vec{x}(E_0)^{\mathbb{N}}\vec{y}$ if at every coordinate $x_n E_0 y_n$.

Theorem 4.1 *(Kechris, Louveau) Assume*

$$E \leq_B E_1.$$

Then exactly one of:

1. $E \leq_B E_0$*; or*

2. $E_1 \leq_B E$*.*

Theorem 4.2 *(Hjorth, Kechris) Assume*

$$E \leq_B (E_0)^{\mathbb{N}}.$$

Then exactly one of:

1. $E \leq_B E_0$*; or*

2. $(E_0)^{\mathbb{N}} \leq_B E$*.*

Admittedly these are far more local in nature.

These are the only immediate successors to E_0 which we have established.

There is an entire spectrum of examples, constructed by Ilijas Farah using ideas from Banach space theory, for which it seems natural to suppose they must be minimal above E_0.

However this remains open, due to problems in the theory of countable Borel equivalence relations which appear unattainable using current techniques.

Moreover Alexander Kechris and Alain Louveau have shown that there is a sense in which there are no more global dichotomy theorems after Harrington, Kechris, Louveau.

5 Anti-structure

Theorem 5.1 *(Louveau, Veličković)*
There are continuum many many $\leq_B$ incomparable Borel equivalence relations.[3]

In fact we can embed $\mathcal{P}(\mathbb{N})$ into $\leq_B$:

There is an assignment

$$S \mapsto E_S$$

of Borel equivalence relations to subsets of $\mathbb{N}$ such that for all $S, T \subset \mathbb{N}$ we have that $T \setminus S$ is finite if and only if

$$E_T \leq_B E_S.$$

Thus there is nothing like the kind of structure for Borel cardinality that one finds with the Wadge degrees.

[3]This first part may have been proved earlier by Hugh Woodin in unpublished work.

Theorem 5.2 *(Kechris, Louveau) There is no Borel $E >_B E_0$ with the property that for all other Borel F we always have one of:*

1. $F \leq_B E$; or

2. $E \leq_B F$.

Two key facts: First of all, Kechris and Louveau showed that E_1 is not Borel reducible to any E_G arising as a result of a continuous Polish group action[4], and secondly Leo Harrington showed that the Borel E_G's of this form are unbounded with respect to Borel reducibility:

Theorem 5.3 *(Harrington) There is a collection $\{E_\alpha : \alpha \in \omega\}$ of Borel equivalence relations such:*

1. Each E_α arises as a result of the continuous Polish group action on a Polish space;

2. For any Borel F there will be some α with E_α not *Borel reducible to F.*

[4]A theoreom due to Howard Becker and Alexander Kechris theorem on changing topologies in the dynamical context shows that there is no basically no difference between equivalence relations induced by continuous actions and induced by Borel actions. *However* it is important that the responsible group be a Polish group – there are certain traces of rigidity for Polish groups, whereas Borel actions of Borel groups can induce any Borel equivalence relation one cares to name

To sketch a proof by contradiction of Kechris and Louveau's result, suppose E was a Borel equivalence relation with the property that for all Borel F we have one of:

1. $F \leq_B E$; or
2. $E \leq_B F$.

Referring back to Harrington's theorem, there will be some α with E_α not Borel reducible to E.

Thus since 1 fails for $F = E_\alpha$ we must have $E <_B E_\alpha$

But E_1 is not Borel reducible to any Polish group action, and hence using the same reasoning we must have $E <_B E_1$.

Which by the Kechris-Louveau dichotomy theorem yields $E \leq_B E_0$.

However this proof prompts the following response:

Question Let E be a Borel equivalence relation. Must we have one of the following:

1. $E \leq_B E_G$ some E_G induced by the continuous action of a Polish group on a Polish space; or
2. $E_1 \leq_B E$?

In other words, is E_1 the *only* obstruction to "classification" or "reduction" to a Polish group action?

At present this is wide open.

The question has, however, been positively answered by Slawomir Solecki in many special cases. In particular, his penetrating structure theorem for Polishable ideals proves it for equivalence relations on $2^{\mathbb{N}}$ arising as the coset equivalence relation of some Borel ideal.

Definition An equivalence relation E on a Polish space X is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$f : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) \cong f(x_2).$$

This notion of classifiability has been subject to close scrutiny, in part since it is so natural from the perspective of a logician.[5]

It might also provide a template of what we could hope to achieve with other notions of classifiability, where some kind of structure theorems can be proved without appeal to a Harrington, Kechris, Louveau type dichotomy theorem.

[5] In fact a Borel equivalence relation E will be $L(\mathbb{R})$ classifiable by countable structures if and only if $|X/E|_{L(\mathbb{R})} \leq |HC|$ – classifiability in this sense amounting to reducible to the hereditarily countable sets

Theorem 6.1 *(Farah) There is a family of continuum many Borel equivalence relations, $(E_r)_{r\in\mathbb{R}}$, such that:*

1. *each E_r is induced by the continuous action of an abelian Polish group on a Polish space; and*
2. *no E_r is classifiable by countable structures;*
3. *for $r \neq s$ the equivalence relations are incomparable with respect to Borel reducibility;*
4. *if $E <_B E_r$, any r, then E is essentially countable, and hence classifiable by countable structures.*

This says that there is no single canonical obstruction to be classifiable by countable in the way we find E_0 as a canonical obstruction to smoothness.

Definition Let G be a Polish group acting continuously on a Polish space X. For V an open neighborhood of 1_G, U an open set containing x, we let

$$O(x, U, V),$$

the U-V-local orbit, be the set of all $\hat{x} \in [x]_G$ such that there is a finite sequence

$$(x_i)_{i \leq k} \subset U$$

such that

$$x_0 = x, \qquad x_k = \hat{x},$$

and each

$$x_{i+1} \in V \cdot x_i.$$

Definition Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

1. every orbit is dense; and
2. every orbit is meager; and
3. for $x \in X$, the local orbits of x are all somewhere dense.

Farah's theorem tells us we can not find even finitely many Borel equivalence relations which are canonical obstructions for classification by countable structures.

Theorem 6.2 *(Hjorth) Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G^X. Assume E_G^X is Borel.*

Then exactly one of:

1. *E_G^X is classifiable by countable structures; or*
2. *G acts turbulently on some Polish space Y and*
$$E_G^Y \leq_B E_G^X.$$

There are in fact cases where one can rule out the existence of turbulent actions by a group, and thus show all the orbit equivalence relations induced by a certain Polish group must be classifiable by countable structures.

Theorem 6.3 *Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G^X. Assume E_G^X is Borel.*

Then exactly one of:

1. *E_G^X is smooth; or*
2. *G acts continuously on a Polish space Y with all orbits dense and meager and*

$$E_G^Y \leq_B E_G^X.$$

Definition If G is a Polish group acting on a Polish space X, we call X *stormy* if for every nonempty open $V \subseteq G$ and $x \in X$ the map

$$V \to [x]_G$$

$$g \mapsto g \cdot x$$

is not an open map.

In a manner parallel to the theory of turbulence stormy provides *the* obstruction for being essentially countable.

7 The wish list

Question Let E be a Borel equivalence relation.

Must we have one of:

1. $E \leq_B E_G$ for some E_G arising as the orbit equivalence relation of a Polish group acting continuously on a Polish space; or
2. $E_1 \leq_B E$?

More generally, if we could establish that there is some analysis of when an equivalence relation is Borel reducible to a Polish group action, then we could lever the theorems regarding turbulence and stormy actions to gain a general understanding of when a Borel equivalence relation admits classification by countable structures or is essentially countable.

Question Let E_G arise from the continuous action of an *abelian* Polish group on a Polish space. Let $E \leq_B E_G$ be a Borel equivalence relation with countable classes.

Must we then have $E \leq_B E_0$?

If so, then Farah's earlier examples would obtain continuum many immediate successors to E_0 in the $\leq_B$ ordering.

Other work of Farah would obtain Borel equivalence relations which are above E_0 but have no immediate successor to E_0 below.

Is there a kind of generalized dichotomy theorem for hyperfiniteness?

Most optimistically:

Question Let E be a countable Borel equivalence relation. Must we have either:

1. $E \leq_B E_0$; or
2. there is a free measure preserving action of $\mathbb{F}_2$ on a standard Borel probability space such that $E_{\mathbb{F}_2} \leq_B E$?

It *is* known that no such $E_{\mathbb{F}_2}$ is Borel reducible to E_0.

This seems wildly optimistic at present, and perhaps it would be less rash to ask it only in the case that E is treeable, but it would in particular have as one of its consequences a positive answer to the following:

Question Let G be a countable *amenable*[6] group. Suppose G acts in a Borel manner on a standard Borel space X.

Must we have $E_G \leq_B E_0$?

The closest result to this is given by a startlingly original combinatorial argument due to Su Gao and Steve Jackson who establish a positive answer in the case G is abelian.

There are no known techniques, or even hints at ideas, which could provide a counterexample to the above question.

[6] *Amenability* can be characterized as the statement that for all $F \subset G$ finite, $\epsilon > 0$, there is some $A \subset G$ finite with

$$\frac{|A \Delta g \cdot A|}{|A|}$$

all $g \in F$.

All known proofs that an equivalence relation is not reducible to E_0 rely on measure theory, and it follows from Connes, Feldman, Weiss that any such E_G must be Borel reducible to E_0 on some conull set with respect to any Borel probability measure.

In fact:

Question Let E be a countable Borel equivalence relation. Are measure theoretic reasons the *only obstruction* to being Borel reducible to E_0?

For instance, if E is countable and not Borel reducible to E_0, must it be the case that there is a Borel probability measure μ such that $E|_A$ is not Borel reducible to E_0 on any conull A?

Any counterexample would require the development of fundamentally new ideas about how to prove some equivalence relations are not $\leq_B$ E_0.