Metric Properties of Sets Definable in $(\mathfrak{R}, \alpha^{-\mathbb{N}})$

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$\mathfrak{R}$ an o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$

o-minimal: definable unary sets are finite unions of intervals

$\operatorname{fr} X = \operatorname{cl} X \setminus X$

$\operatorname{Vol}_d(X)$ = d-dimensional volume of X

Fact. Let X be bounded and definable in $\mathfrak{R}$ such that $\dim X = d$. Then $\operatorname{Vol}_d(X) < \infty$ and $\dim \operatorname{fr} X < d$.

There are embedded submanifolds of $\mathbb{R}^n$ for which this fails:

(1) $y = \sin(1/x)$ fr has dim = 1, curve has **infinite** length

(2) $y = x\sin(1/x)$ fr has dim = 0, curve has **infinite** length

(3) $y = x^2\sin(1/x)$ fr has dim = 0, curve has **finite** length

For fixed $\alpha > 1$, consider expansion of $\mathfrak{R}$ by

$$\alpha^{-\mathbb{N}} := \{\alpha^{-n} : n \in \mathbb{N}\}$$

1. **Theorem.** *Suppose* $\mathfrak{R}$ *defines no irrational power functions on* $\mathbb{R}^{>0}$*. Let* X *be bounded and definable in* $(\mathfrak{R}, \alpha^{-\mathbb{N}})$ *such that* $\dim X = d$*. Then*

$$\mathrm{Vol}_d\, X < \infty \text{ \textit{iff} } \dim \mathrm{fr}\, \mathrm{reg}\, X < d$$

In particular, if X *is an embedded submanifold of some* $\mathbb{R}^n$*, then*

$$\mathrm{Vol}_d\, X < \infty \text{ \textit{iff} } \dim \mathrm{fr}\, X < d$$

.

Thus definable sets in $(\mathfrak{R}, \alpha^{-\mathbb{N}})$ cannot behave as curve (2).

Theorem 1 is optimal in two senses:

- Behavior as in (1) is unavoidable: polygonal path connecting points $(x^2, 1)$ to points $(\alpha x^2, -1)$ and $(\alpha^{-1}x^2, -1)$ for $x \in \alpha^{-\mathbb{N}}$ ($\alpha^{-\mathbb{N}}$ itself is a $\dim = 0$ example)

- If $\mathfrak{R}$ defines an irrational power function on $\mathbb{R}^{>0}$, then a result of Hieronymi shows that $(\mathfrak{R}, \alpha^{-\mathbb{N}})$ defines $\mathbb{Z}$, so this requirement for Theorem 1 cannot be avoided.

Outline of proof of Theorem 1. For $x \in \mathbb{R}$, let

$$\lambda(x) = \begin{cases} 0 & : x \leq 0 \\ \alpha^n & : n \in \mathbb{Z} \text{ and } \alpha^n \leq x < \alpha^{n+1} \end{cases}$$

Then $(\mathfrak{R}, \alpha^{-\mathbb{N}})$ is interdefinable with $(\mathfrak{R}, \lambda)$ and

2. **Theorem** (Miller)**.** *The theory of $(\mathfrak{R}, \lambda)$ admits QE and is $\forall$-axiomatizable relative to the theory of $\mathfrak{R}$.*

Apply o-minimal trivialization and definable choice in $(\mathfrak{R}, \alpha^{-\mathbb{N}})$:

3. **Lemma.** *Let* $X \subseteq \mathbb{R}^n$ *be definable in* $(\mathfrak{R}, \alpha^{-\mathbb{N}})$. *Then* X *is a finite union of images* $F((\alpha^{-\mathbb{N}})^m \times [0,1]^d)$, *where* F *is definable in* $\mathfrak{R}$ *and injective on its support intersected with* $(\alpha^{-\mathbb{N}})^m \times [0,1]^d$.

Combine Lemma with Cluckers' decomposition theorem for definable sets in Presburger arithmetic and the polynomially-bounded preparation theorem of van den Dries and Speissegger for induced structure:

4. **Theorem.** *Let* $Z \subseteq (\alpha^{-\mathbb{N}})^m$ *be definable in* $(\mathfrak{R}, \alpha^{-\mathbb{N}})$. *Then* Z *is definable in* $(\mathbb{R}, \cdot, <)$.

Volume estimates: Using Pawłucki's Lipschitz cell decomposition theorem and induced structure, reduce to the case that the derivative of F over the last d coordinates $D_x F(h,x)$ is triangular and the volume element $|\det D_x F(h,x)|$ is uniformly Lipschitz. Put

$$\mathrm{A}(h) := \mathrm{vol}_d(F(\{h\} \times [0,1]^d))$$

$$= \int |\det D_x F(h,x)| dx$$

A is uniformly Lipschitz, and so has continuous extension to the boundary of its support

Estimate the integral definably to get dfbl (in $\mathfrak{R}$) functions $V, U : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $U(h) < A(h) < V(h)$ and

$$0 < U(h) \text{ if } 0 < A(h)$$

$$\lim_{h \to z} V(h) = 0 \text{ if } \lim_{h \to z} A(h) = 0$$

To determine when the volume of $F((\alpha^{-\mathbb{N}})^m \times [0,1]^d)$ is finite, we sum the values of these functions over $(\alpha^{-\mathbb{N}})^m$ to approximate.

5. **Lemma.** *Let* $V : \mathbb{R}^m \to \mathbb{R}^{\geq 0}$ *be definable in* $\mathfrak{R}$ *such that* $\lim_{h \to z,\, h \in (\alpha^{-\mathbb{N}})^m} V(h) = 0$ *for each* $z \in \mathrm{fr}(\alpha^{-\mathbb{N}})^m$. *Then* $\sum_{h \in (\alpha^{-\mathbb{N}})^m} V(h) < \infty$.

The proof is based on asymptotics provided by [DS]-preparation and the induced structure result above.

Final steps in the argument show that $A(h)$ goes to 0 on $(\alpha^{-\mathbb{N}})^m$ iff the dimension of frontier of the set $X = F((\alpha^{-\mathbb{N}})^m \times [0,1]^d)$ is less than d.

If $A(h)$ goes to 0 on $\mathrm{fr}(\alpha^{-\mathbb{N}})^m$, then apply Cauchy-Binet to see that the frontier of X is the union of the frontiers of the sets $\{F(\{h\} \times [0,1]^d) : h \in \alpha^{-\mathbb{N}}\}$ and the frontier of the union of family of lower dimensional subsets.

The frontier of X thus has dimension $< d$ as a consequence of Miller's regular manifold decomposition for sets definable in $(\mathfrak{R}, \alpha^{-\mathbb{N}})$.

If $A(h)$ does not go to 0 on $(\alpha^{-\mathbb{N}})^m$, then use the lower estimate U, induced structure, and uniform Lipschitz for F to show that the dimension of the frontier must be at least d.

Consequence of the proof:

6. **Theorem.** *Let* $S \subseteq \mathbb{R}^{\geq 0}$ *be discrete and definable in* $(\mathfrak{R}, \alpha^{-\mathbb{N}})$*. Then* $\sum_{s \in S} s < \infty$ *iff* S *is bounded and its only limit point is* 0*.*

Extension to other sequences: Miller and Tyne proved a result similar to Theorem 2 for certain classes of iteration sequences, and results here go through for these structures (more easily in fact; simpler induced structure).

An interesting consequence of Theorem 1 is:

7. **Proposition.** *Let* $\{X_y : y \in Y\}$ *be a family of bounded sets definable in* $(\mathfrak{R}, \alpha^{-\mathbb{N}})$. *Then the set of all* y *such that* $\mathrm{Vol}_d\, X_y < \infty$ *is definable.*

Work underway to generalize results to the unbounded case.

FURTHER WORK

Stratification theory for definable sets in $(\mathfrak{R}, \alpha^{-\mathbb{N}})$ where m becomes a complexity parameter analogous to Cantor-Bendixson rank.

What closed definable sets are 0-sets of definable C^p functions?