Automorphisms of Short Saturated Models of PA

Ermek Nurkhaidarov (Penn State) &
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A model M is *boundedly saturated* iff for every $A \subseteq M$ with $|A| < |M|$, M realizes every bounded type with parameters from A.

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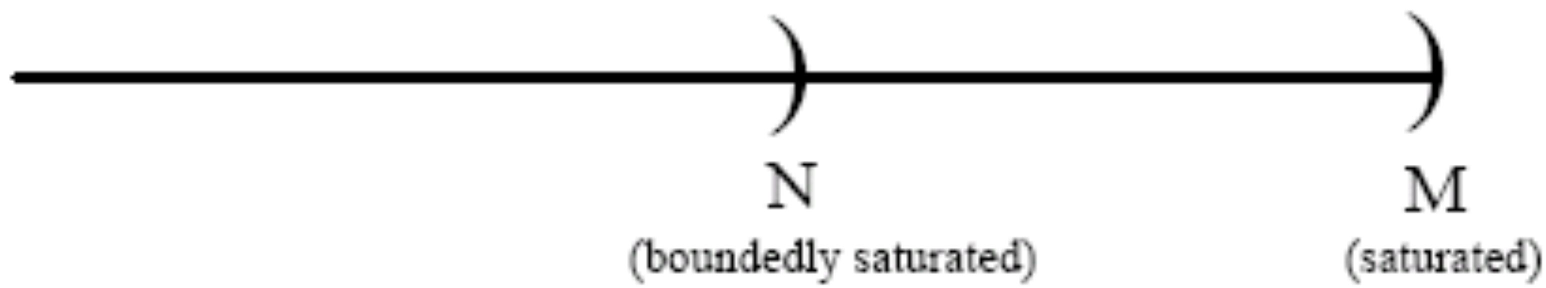

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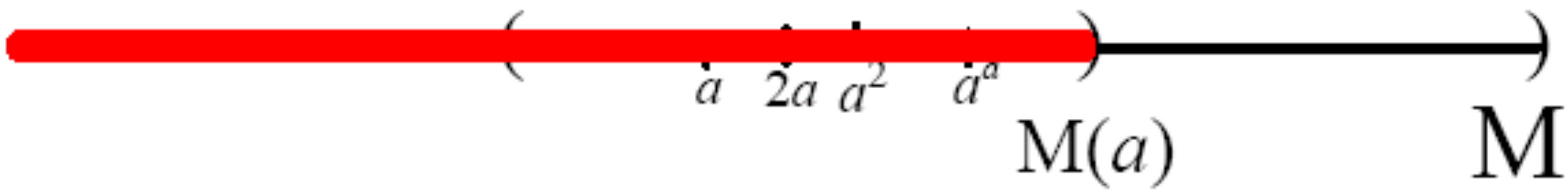

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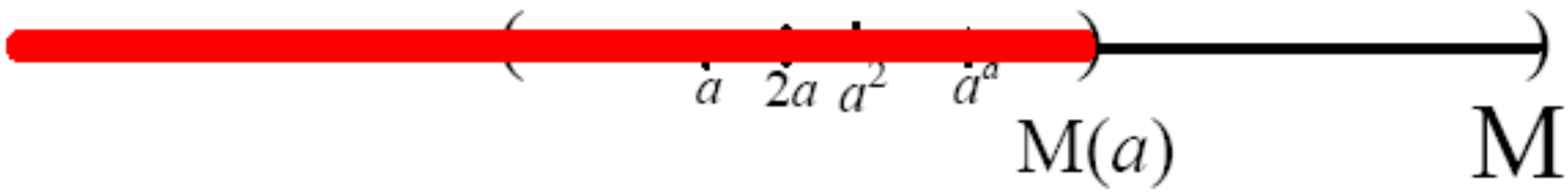

A model M is *short* if $M = M(a)$ for some $a \in M$.

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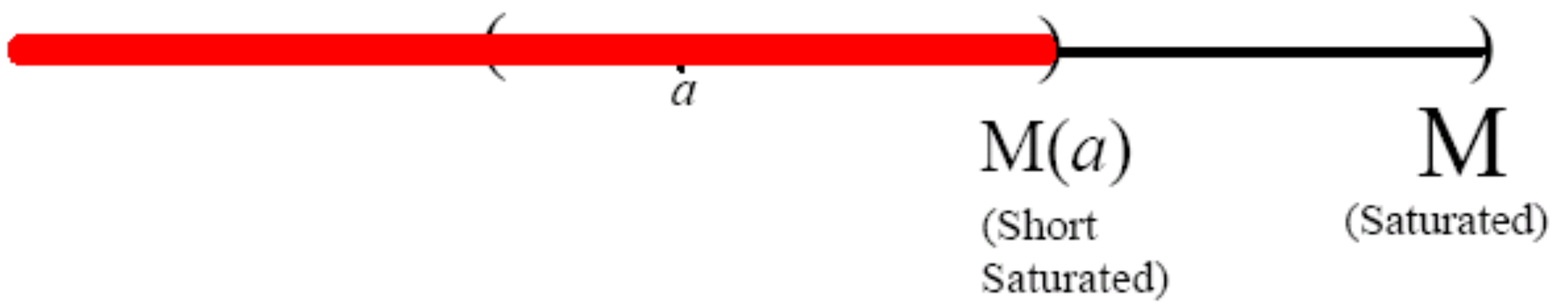

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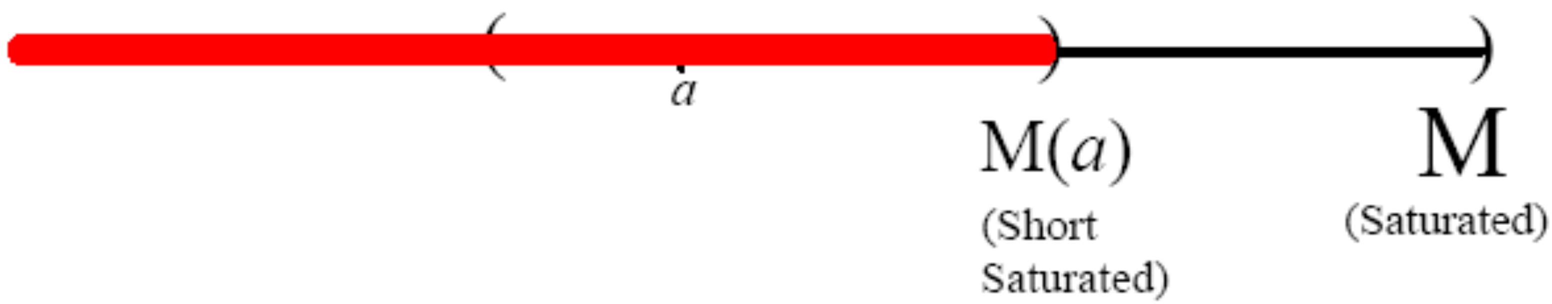
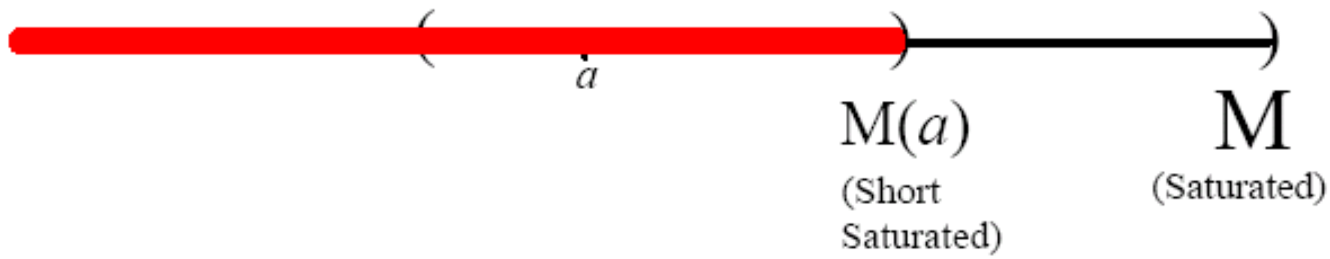


For the rest of the talk we fix M a saturated model of PA and $M(a)$ a short saturated elementary initial segment of M.

Automorphisms of Short Saturated Models of PA



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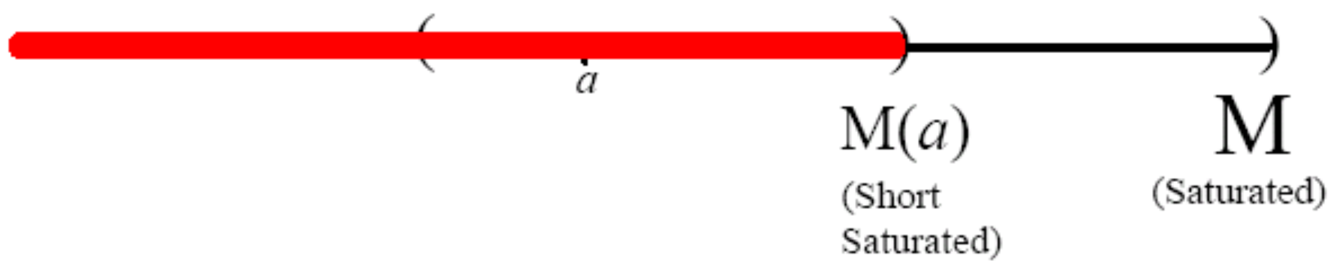

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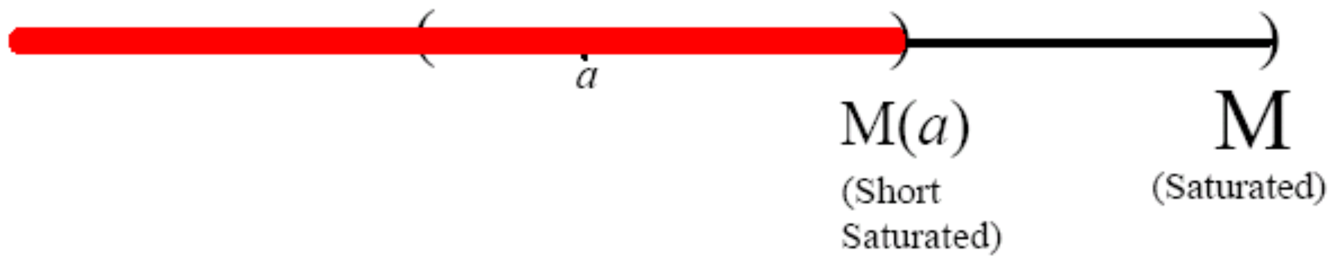

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In fact, any automorphism of any boundedly saturated elementary initial segment of M which is not saturated can be extended to an automorphism of M.

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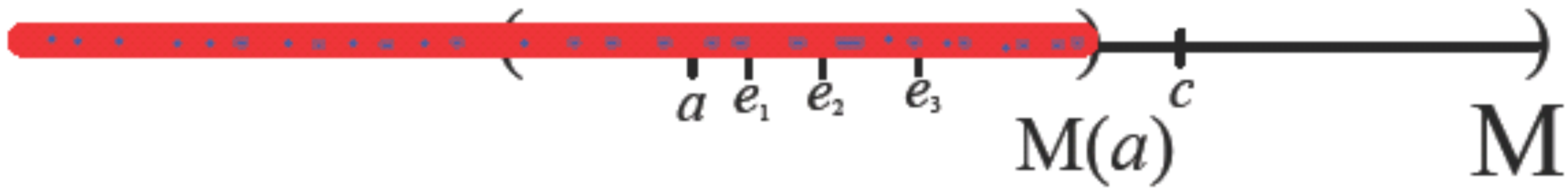
- Remains to show:

  Lemma: Every automorphism of a short saturated model sends coded sets to coded sets.

Sketch of Proof of Lemma

- X (subset of M(a) coded in M by c)



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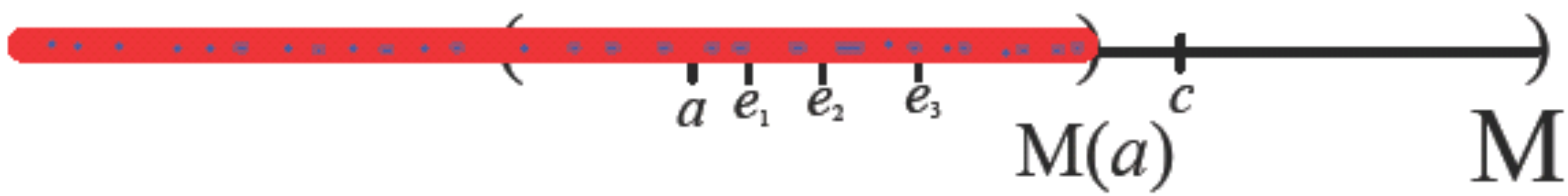

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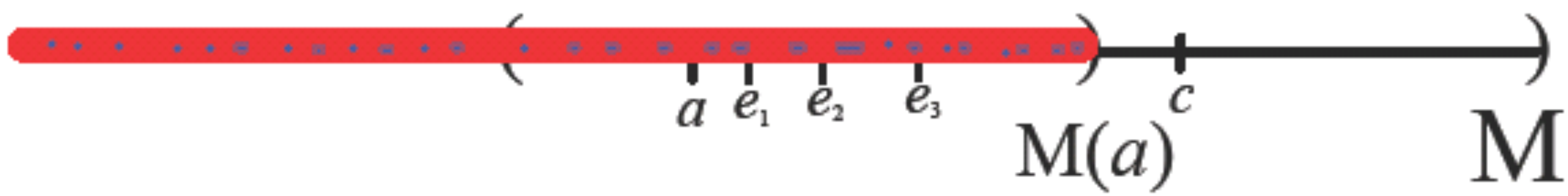

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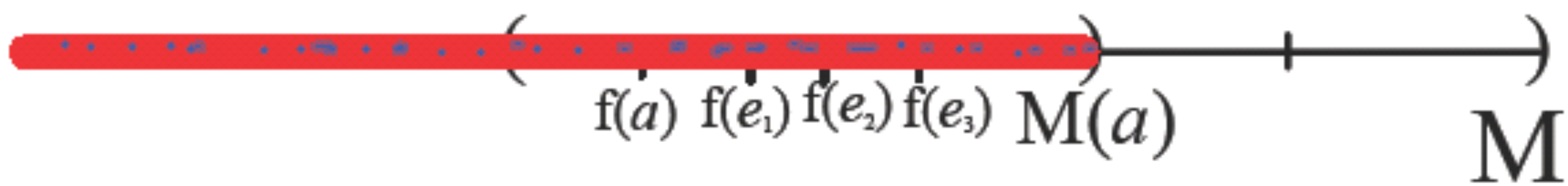

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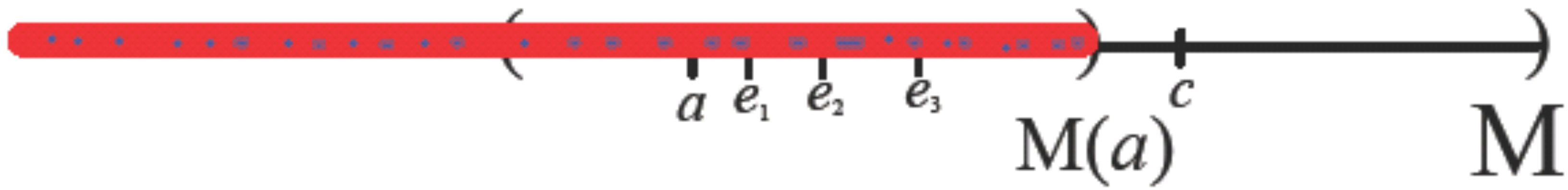

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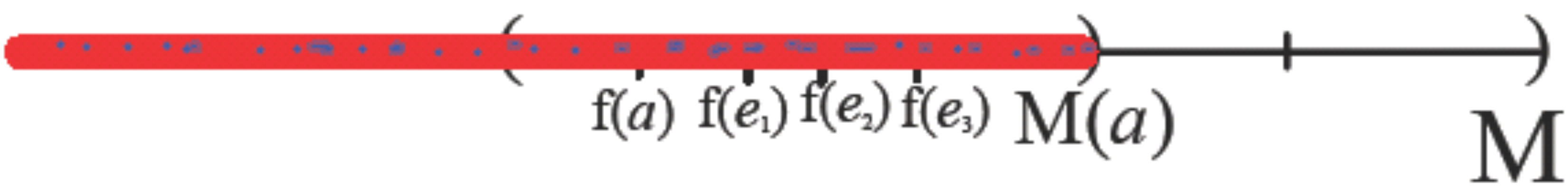

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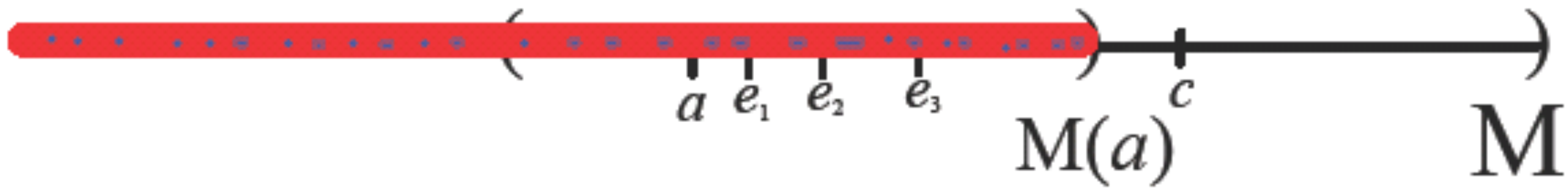

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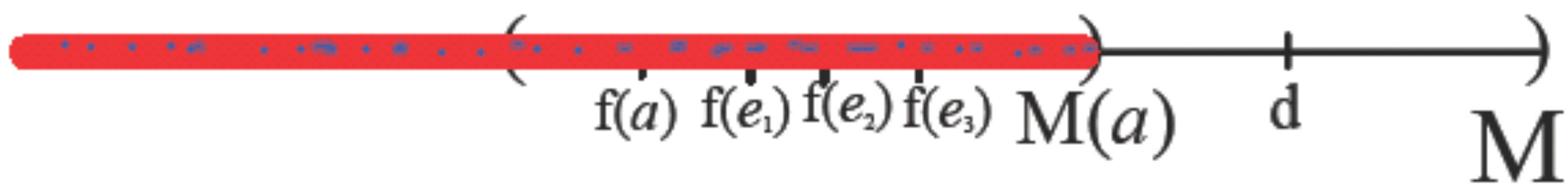

f(e_1), f(e_2), f(e_3), ... is a sequence of elements coding initial segments of f(X).
There is an element d in M such the first lh(f(e_i)) elements coded by d are exactly the elements of f(e_i) for every $i \in \omega$

Main Results

Theorem: Let M be a saturated model of Peano Arithmetic. Then H is a closed normal subgroup of $\mathrm{Aut}(M)$ iff there exists an invariant cut $I \subseteq M$ such that $H = \mathrm{Aut}(M)_{(I)}$.

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Corollary: Let M be a saturated model of Peano Arithmetic. Then there are short saturated elementary initial segments of M, $M(a)$ and $M(b)$ whose automorphism groups are non-isomorphic as topological groups.

- All these results and more are to appear in a future issue of the Notre Dame Journal of Formal Logic, in the paper:

Automorphisms of saturated and boundedly saturated models of arithmetic

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Thank You!