GEOMETRIC AXIOMS FOR THE THEORY $\mathrm{DCF}_{0,m+1}$

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Motivation

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- In the 50's Robinson showed that the class of existentially closed ordinary differential fields (of characteristic zero) is elementary. Then, in the 70's, Blum gave elegant algebraic axioms:

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- In 1998, Pierce and Pillay gave axioms of DCF_0 in terms of algebraic varieties and their prolongation: $K \models ACF_0$ and

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- Geometric axiomatizations have been given for other theories $ACFA$, $DCFA_0$, DCF_p and $SCH_{p,e}$.

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do not axiomatize $DCF_{0,m}$.
- Nonetheless, in 2010, Pierce formulated geometric axioms in arbitrary characteristic.

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- In other words, we characterize $DCF_{0,m+1}$ in terms of the geometry of $DCF_{0,m}$.

Theorem

$(K, \Delta, D) \models DCF_{0,m+1}$ if and only if

1. $(K, \Delta) \models DCF_{0,m}$
2. For each pair of irreducible Δ-closed sets V and W such that $W \subseteq \tau_{D/\Delta} V$ and W projects Δ-dominantly onto V, there is a K-point $\bar{a} \in V$ such that $(\bar{a}, D\bar{a}) \in W$.

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- $\theta\bar{x} = (\theta_1\bar{x}, \theta_2\bar{x}, \ldots)$ the set of algebraic indeterminates $\delta_m^{r_m} \cdots \delta_1^{r_1} x_i$, ordered w.r.t. the canonical ranking.

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- For $f \in K\{\bar{x}\}$, the *Jacobian*

$$df(\bar{x}) := \left(\frac{\partial f}{\partial \theta_1 \bar{x}}(\bar{x}), \frac{\partial f}{\partial \theta_2 \bar{x}}(\bar{x}), \dots, \frac{\partial f}{\partial \theta_h \bar{x}}(\bar{x}), 0, 0, \dots\right).$$

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- $D:K\to K$ another derivation commuting with Δ.
- f^D the Δ-polynomial obtained by applying D to the coefficients of f.

Definition

Let $\tau_{D/\Delta} : K\{\bar{x}\} \to K\{\bar{x}, \bar{y}\}$ be

$$\tau_{D/\Delta} f(\bar{x}, \bar{y}) = df(\bar{x}) \cdot \theta\bar{y} + f^D(\bar{x})$$

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Definition of D/Δ-prolongation

Let $V \subseteq K^n$ be a Δ-closed set, then $\tau_{D/\Delta} V \subseteq K^{2n}$ is the Δ-closed set

$$\tau_{D/\Delta} V = \mathcal{V}(f, \tau_{D/\Delta} f : f \in \mathcal{I}(V/K)) \tag{1}$$

$\mathcal{I}(V/K) := \{f \in K\{\bar{x}\} : f(V) = 0\}$.

Does $\tau_{D/\Delta} V$ vary uniformly with V?
If $\mathcal{I}(V/K)$ is differentially generated by $f_1, \ldots, f_s$ then one only needs to check equation (1) for the f_i's.

Characterization of $\mathrm{DCF}_{0,m+1}$

Theorem 1 (L.S.)

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- Containment in $\tau_{D/\Delta} V$?
- Δ-dominant projections?

Pierce-Pillay Axioms

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However, we do not need so much. In fact, the Pierce-Pillay axioms hold even if one removes the word irreducibility and replace dominance by surjectivity.

In the case of several derivations we can almost do the same.

We remove the irreducibility hypothesis using

If X is a K-irreducible component of V then the fibres of $\tau_{D/\Delta}X$ and $\tau_{D/\Delta}V$ are generically the same.

To deal with containments in $\tau_{D/\Delta}V$ we have

Suppose $(K,\Delta) \models DCF_{0,m}$. If $V = \mathcal{V}(f_1,\dots,f_s)$, then

$$\tau_{D/\Delta}V = \mathcal{V}(f_1,\dots,f_s,\tau_{D/\Delta}f_1,\dots,\tau_{D/\Delta}f_s)$$

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Δ-dominance?

In case $m=0$ we can replace dominance by surjectivity. This follows from the fact that if a is D-algebraic then $D^{k+1}a$ is in $K(a,Da,\dots,D^ka)$, for some k. This is not true with several derivations!

For every $M \in GL_{m+1}(\mathbb{Q})$, let $\bar{\Delta} = \{\bar{\delta}_1, \ldots, \bar{\delta}_m\}$ and $\bar{D}$ be the derivations defined by

$$\begin{pmatrix} \bar{\delta}_1 \\ \vdots \\ \bar{\delta}_m \\ \bar{D} \end{pmatrix} = M \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_m \\ D \end{pmatrix}$$

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Theorem (Kolchin)

If a is (Δ, D)-algebraic over K, then there is k and a matrix $M \in GL_{m+1}(\mathbb{Q})$ such that, writing $(\bar{\Delta}, \bar{D}) = M(\Delta, D)$, we have that $\bar{D}^{k+1}a$ is in the $\bar{\Delta}$-field generated by $a, \bar{D}a, \dots \bar{D}^k a$.

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Condition (2) is indeed first order. Expressible by infinitely many sentences, one for each choice of M, $f_1, \dots, f_s$ and shape of W.