Boolean Subalgebras and Computable Copies

Russell Miller

Queens College & CUNY Graduate Center, New York, NY.

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Spectra of Structures and Relations

Defns.

The *spectrum* of a countable structure $\mathcal{S}$ is the set

$$\mathrm{Spec}(\mathcal{S}) = \{\deg(\mathcal{M}) : \mathcal{M} \cong \mathcal{S} \ \&\ \mathrm{dom}(\mathcal{M}) = \omega\}.$$

Let R be a relation on a computable structure $\mathcal{B}$. The *spectrum of R* (as a relation on $\mathcal{B}$) is the set

$$\mathrm{DgSp}_{\mathcal{B}}(R) = \{\deg(Q) : \exists \text{ computable } \mathcal{C} \text{ with } (\mathcal{C}, Q) \cong (\mathcal{B}, R)\}.$$

We focus on spectra of unary relations (equivalently, suborders) on the computable dense linear order $\mathbb{Q}$, and spectra of Boolean subalgebras of the computable atomless Boolean algebra $\mathcal{B}$. Both these structures are computably ultrahomogeneous and universal for countable models.

Facts for Linear Orders

Theorem (Frolov, Harizanov, Kalimullin, Kudinov, & Miller 2011)

There exists a relation R on $\mathbb{Q}$ such that

$$\mathrm{DgSp}_{\mathbb{Q}}(R) = \{\boldsymbol{d} : \boldsymbol{d}' \geq \mathbf{0}''\}.$$

However, by a result of Knight from 1986, this set is not the spectrum of any linear order.

The converse is impossible: all spectra of linear orders are spectra of unary relations on $\mathbb{Q}$, by a theorem of Harizanov & Miller (2007).

Theorem (FHKKM 2011)

There exists a relation U on $\mathbb{Q}$ such that $\mathrm{DgSp}_{\mathbb{Q}}(U) = \{\boldsymbol{d} : \boldsymbol{d}' > \mathbf{0}'\}$.

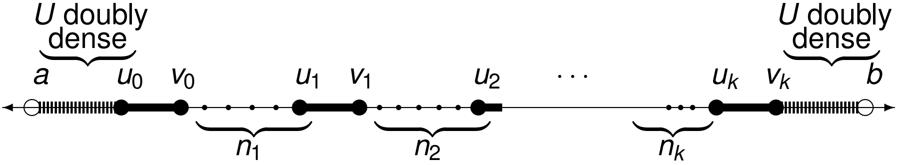
It is unknown whether $\{\boldsymbol{d} : \boldsymbol{d}' > \mathbf{0}'\}$ can be the spectrum of a LO.

Construction for the FHKKM Thm.

By a result of Wehner, for each set $C \subseteq \omega$, there exists a family $\mathbb{F}$ of finite sets such that for all D:

$$\mathbb{F} \text{ has an enumeration uniformly computable in } D \iff D >_T C.$$

For a single finite set $F = \{n_1, n_2, \ldots, n_k\}$, we code F into a relation $U = U_{F,a,b}$ on the interval $[a, b]$ of $\mathbb{Q}$:



“Doubly dense” means that both U and its complement are dense in that subinterval.

Low$_n$ Boolean Algebras

Let $\mathcal{A}$ be a Boolean algebra, with domain ω.

Theorems

- If Spec($\mathcal{A}$) contains a low degree, then it contains the degree **0** (Downey-Jockusch).
- If Spec($\mathcal{A}$) contains a low$_2$ degree, then it contains the degree **0** (Thurber).
- If Spec($\mathcal{A}$) contains a low$_3$ or low$_4$ degree, then it contains the degree **0** (Knight-Stob).

It remains open whether this holds for low$_5$ Boolean algebras. By work of Harris and Montalbán, this problem is quantifiably more difficult.

Question: Do analogous results hold for spectra of Boolean subalgebras of the computable atomless Boolean algebra $\mathcal{B}$?

Facts about Boolean Algebras

- The computable atomless Boolean algebra $\mathcal{B}$ is often represented as the BA of (finite unions of) intervals $[a, b)$ in $\mathbb{Q}$ under $\cup$ and $\cap$. (We include the intervals $(-\infty, b)$ and $[a, +\infty)$.)
- This $\mathcal{B}$ is *spectrally universal* for BA's, just as $\mathbb{Q}$ is for linear orders. (Csima, Harizanov, M., Montalbán.)
- Using results of Jockusch & Soare, H&M showed that there exists a unary relation R on $\mathcal{B}$ whose spectrum contains a low degree, but not $\mathbf{0}$. However, this R is not a Boolean subalgebra. Montalbán asked whether the same can be done for a Boolean subalgebra.

Double Density and $\mathcal{A}$-atoms for Boolean Algebras

The key to the FHKKM theorem was the ambient structure $\mathbb{Q}$, and the notion of *double density*: both U and its complement can be dense in the same interval in $\mathbb{Q}$.

Defn.

Let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{B}$. $\mathcal{A}$ is *doubly dense* within $\mathcal{B}$ if, for every finite Boolean subalgebra $\mathcal{B}_0 \subseteq \mathcal{B}$, $(\mathcal{B}, \mathcal{A})$ realizes every possible finite extension of $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0)$ to a larger BA with Boolean subalgebra.

For a nonempty $x \in \mathcal{B}$, we say that $\mathcal{A}$ is *doubly dense within* x if $x \in \mathcal{A}$ and $\mathcal{A}_x = \{a \in \mathcal{A} : a \subseteq x\}$ is doubly dense within the induced atomless Boolean algebra $\mathcal{B}_x = \{y \in \mathcal{B} : y \subseteq x\}$.

Defn.

An $x \in \mathcal{B}$ is an $\mathcal{A}$*-atom* if $x \in \mathcal{A}$ and $x \neq \emptyset$ and $\mathcal{A}_x = \{\emptyset, x\}$.

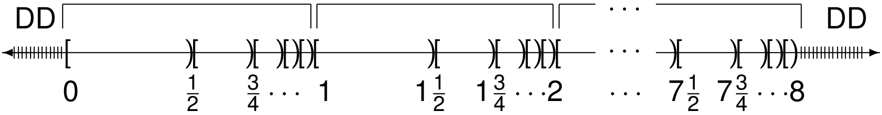
It is $\Pi_1^{\mathcal{A}}$ whether a given $x \in \mathcal{B}$ is an $\mathcal{A}$-atom, and $\Pi_2^{\mathcal{A}}$ whether $\mathcal{A}$ is doubly dense within a given $x \in \mathcal{B}$.

Coding a Fourth Jump $C^{(4)}$ into $\mathcal{A}$

Now we build a specific Boolean subalgebra $\mathcal{A}$ of $\mathcal{B}$.
Let $C^{(4)} = \{n_0 < n_1 < n_2 < \cdots\}$. We first code n_0 into $\mathcal{A}$ as follows:

- Subdivide $[0,1)$ into subintervals $[0, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), \ldots$, and put all these subintervals (but *not* $[0,1)$ itself) into $\mathcal{A}$.
- Do the same with $[1,2)$, then $[2,3)$, up to $[2^{n_0} - 1, 2^{n_0})$.
- Put $[0, 2^{n_0})$ into $\mathcal{A}$.
- Make $\mathcal{A}$ doubly dense within $[2^{n_0}, 2^{n_0} + 1)$.
- Go on to n_1, putting $[2^{n_0} + 1, 2^{n_0} + 1 + 2^{n_1})$ into $\mathcal{A}$, etc.

We also make $\mathcal{A}$ doubly dense within $(-\infty, 0)$, and close $\mathcal{A}$ under complements and finite unions. Thus $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{B}$.



Our construction causes $\mathcal{A}$ to contain a 2^n*-fold* $\mathcal{A}$*-supremum* for precisely those n which lie in $C^{(4)}$. (In this picture, $n_0 = 3$.)

$\mathcal{A}$-suprema

Defn.

An element $x \in \mathcal{B}$ is an $\mathcal{A}$-*supremum* if x is the least upper bound in $\mathcal{B}$ of an infinite set of $\mathcal{A}$-atoms.
Such an x is a *single* $\mathcal{A}$-*supremum* if x is not the union of two disjoint $\mathcal{A}$-suprema.
Finally, $x \in \mathcal{B}$ is a k-*fold* $\mathcal{A}$-*supremum* if x is the union of k disjoint single $\mathcal{A}$-suprema.

The property of being a single $\mathcal{A}$-supremum is $\Pi_3^{\mathcal{A}}$: it holds iff:

- $\mathcal{A}$ is not doubly dense within any $y \subseteq x$; and
- x contains infinitely many $\mathcal{A}$-atoms; and
- every $\mathcal{A}$-atom a has either $a \subseteq x$ or $a \cap x = \emptyset$; and
- $(\forall y \in \mathcal{B})$[either $x \cap y$ or $x - y$ is contained in a finite union of $\mathcal{A}$-atoms].

So the property of being a k-fold $\mathcal{A}$-supremum is $\Sigma_4^{\mathcal{A}}$, uniformly in k.

Decoding $C^{(4)}$ from $\mathcal{A}$

The idea is that $n \in C^{(4)}$ iff $\mathcal{A}$ contains a 2^n-fold $\mathcal{A}$-supremum. This property is $\Sigma_4^{\mathcal{A}}$. Therefore, if C is not low_4, then $C^{(4)} \not\leq \emptyset^{(4)}$, and there can be no computable $\tilde{A} \subseteq \mathcal{B}$ with $(\mathcal{B}, \tilde{A}) \cong (\mathcal{B}, \mathcal{A})$.

We claim that, for every C, the process above builds a Boolean subalgebra $\mathcal{A}$ such that $\deg(C) \in \text{DgSp}_{\mathcal{B}}(\mathcal{A})$. By taking C to be low_5 but not low_4, this will prove:

Theorem (M., 2011)

There exists a Boolean subalgebra $\mathcal{A}$ of the computable atomless BA $\mathcal{B}$ such that $\text{DgSp}_{\mathcal{B}}(\mathcal{A})$ contains a low_5 degree, but not $\mathbf{0}$.

$\mathcal{A}$ as a Boolean Algebra

Just as with linear orders, this construction used the ambient structure $\mathcal{B}$ in an essential way. If we regard $\mathcal{A}$ as a BA in its own right, then all k-fold $\mathcal{A}$-suprema turn into single $\mathcal{A}$-suprema, and the coding of $C^{(4)}$ vanishes. Indeed, this $\mathcal{A}$ has a computable copy. So the question remains:

Question

Does there exist a Boolean algebra whose spectrum contains a low$_5$ degree, but does not contain $\mathbf{0}$?

Further Questions

Another question is the subject of current work by R. Steiner:

Question

Do all Boolean subalgebras $\mathcal{A} \subseteq \mathcal{B}$ for which $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$ contains a low_4 degree also have computable copies? (Steiner's conjecture: No.) If not, then how about low_3, low_2, and low?

A negative answer to either question would give an example of a set of Turing degrees which is the spectrum of a Boolean subalgebra of $\mathcal{B}$, but not of any Boolean algebra (as a structure), and would thus prove that for BA's, the ambient structure does enable extra information content. For BA's, it remains open whether this is possible. For LO's, the ambient structure $\mathbb{Q}$ does allow extra information to be coded, but for graphs, the random graph as ambient structure does not allow any information which could not already have been coded into some countable graph.

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