Separating club guessing principles

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(+) is the statement that there exists a stationary set $\mathcal{S}$ consisting of countable elementary substructures of $H(\aleph_2)$, such that for all X, Y in $\mathcal{S}$, if

$$X \cap \omega_1 = Y \cap \omega_1$$

and $C \in X$, $D \in Y$ are club subsets of ω_1, then

$$C \cap D \cap X \cap \omega_1$$

is nonempty.

If there exist a proper class of Woodin cardinals, then it is possible to force $\neg(+)$ + “every club contains the indiscernibles of some real” without changing the theory of $L(\mathbb{R})$.

It follows that if there exist proper class many Woodin cardinals, then there exist reals x, y such that the least common indiscernible of x and y is the least common indiscernible of $x^{\#}$ and $y^{\#}$.

Theorem 1 (Moore). *MRP* $\Rightarrow \neg(+)$

As reformulated by Moore, (+) is equivalent to the statement that there exist

$$\mathcal{F}_\alpha \, (\alpha < \omega_1)$$

such that

- each $\mathcal{F}_\alpha$ consists of club subsets of α having pairwise cofinal intersection,

- for every club $C \subseteq \omega_1$, $C \cap \alpha \in \mathcal{F}_\alpha$ for stationarily many α.

$(+)_n$: the elements of each $\mathcal{F}_\alpha$ have n-wise cofinal intersection

$(+)_{<\omega}$: each $\mathcal{F}_\alpha$ is closed under intersections

$(+)^c$: for each club $C \subseteq \omega_1$, $C \cap \alpha \in \mathcal{F}_\alpha$ for club many α

Club Guessing is the statement that there exist

$$a_\alpha \, (\alpha < \omega_1)$$

such that

- each a_α is a cofinal subset of α;
- for each club $C \subseteq \omega_1$, $a_\alpha \subseteq C$ for stationarily many α.

Strong Club Guessing is the statement that there exist

$$a_\alpha\,(\alpha < \omega_1)$$

such that

- each a_α is a cofinal subset of α;
- for each club $C \subseteq \omega_1$, $a_\alpha \setminus C$ is finite for club many α.

Club Guessing implies $(+)_{<\omega}$

2 Question. Is is (ever/always) possible to force the existence of a Club Guessing sequence with a c.c.c. forcing?

If $(+)_{<\omega}$ holds, yes.

If a Strong Club Guessing can be forced by a c.c.c. forcing, then $(+)^{c}_{<\omega}$ holds.

Using this, one can show that assuming MRP + "NS_{ω_1} is saturated" that is not possible to force Club Guessing with a c.c.c. forcing.

Theorem 3. $\neg(+)$ *is consistent with GCH.*

Let $P(F)$ be the standard forcing to destroy a witness F to $(+)$. Applying results of Shelah and Moore, showing that a countable support iteration of forcings of the form $P(F)$ does not add reals requires showing :

- $P(F)$ is proper in all proper forcing extensions

- whenever M is the transitive collapse of a suitable elementary submodel, and M embeds elementarily into models N_1 and N_2, there is (below any condition) an M-generic filter g which is bounded below in any model which contains g as an element and which emebds either of N_1 and N_2 elementarily.

$\diamondsuit^+$ is the statement that there exists

$$\mathcal{A}_\alpha \, (\alpha < \omega_1)$$

such that

- each $\mathcal{A}_\alpha$ is a countable collection of subsets of α,

- for each $X \subseteq \omega_1$ there is a club $D \subseteq \omega_1$ such that

  $$\{X \cap \alpha, D \cap \alpha\} \subseteq \mathcal{A}_\alpha$$

  for all $\alpha \in D$.

Theorem 4. $\diamondsuit^+$ *does not imply* $(+)^c$ *(and therefore does not imply Strong Club Guessing).*

Start with a model of GCH $+ \neg(+)$ and use the usual partial order for forcing $\diamondsuit^+$. Given two elementary submodels witnessing a failure of $(+)$, find a filter which is generic for both of them.

Theorem 5. *CH + $(+)_{<\omega}$ does not imply Club Guessing.*

6 Definition. A $(+)_{<\omega}$-sequence $\langle f_\alpha : \alpha < \omega \rangle$ is *p-point like* if for each $\subseteq$-descending sequence from each f_α there is a member of f_α mod-finite contained in each member of the sequence.

Such sequences exist if Club Guessing holds and there is a p-point.

Lemma 7. *If F is a p-point like $(+)_{<\omega}$-sequence, and P is a $(+)_{<\omega}$-proper forcing for F, then F generates a $(+)_{<\omega}$-sequence in the P-extension.*

Lemma 8. *If F is a p-point like $(+)_{<\omega}$-sequence, then any countable support iteration of the standard forcing to destroy Club Guessing sequences is $(+)_{<\omega}$-proper and adds no new reals.*

Trying with $\mathbb{P}_{\text{max}}$......

Basic problem: given a model M with a $(+)_{<\omega}$-sequence

$$F = \langle f_\alpha : \alpha < \omega_1^M \rangle,$$

produce an iteration

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \beta < \omega_1, \alpha \leq \gamma \leq \omega_1 \rangle$$

of M so that $j_{0,\omega_1}(F)$ can be expanded to a $(+)_{<\omega}$-sequence (while destroying any given Club Guessing sequence from M).

Weak condensation for $H(\aleph_2)$ is the statement that there exist $\mathcal{N}_\alpha$ $(\alpha < \omega_1)$ such that for each $\alpha < \omega_1$,

- $\mathcal{N}_\alpha$ is a countable set of transitive collapses of countable elementary submodels of $H(\aleph_2)$ which is linearly ordered by $\subseteq$,

- for all $N \in \mathcal{N}_\alpha$, $\omega_1^N = \alpha$,

and such that for club many countable elementary submodels X of $H(\aleph_2)$, the transitive collapse of X is an element of $\mathcal{N}_{X \cap \omega_1}$.

Theorem 9 (Woodin)**.** *Assuming AD, there are inner models with Woodin cardinals satisfying weak condensation for* $H(\aleph_2)$.

Theorem 10. *Assuming the consistency of AD, there is a model of* $(+)^c_{<\omega}$ *in which Club Guessing fails.*

Club weak club guessing is the statement that there is a sequence

$$\langle a_\alpha : \alpha < \omega_1 \rangle$$

such that each α_α is a cofinal subset of α of ordertype at most ω, and such that for every club $C \subseteq \omega_1$,

$$a_\alpha \cap C$$

is infinite for club many α.

The *Interval Hitting Principle* is the statement that there exists a set

$$\{b_\alpha : \alpha < \omega_1\}$$

such that each b_α is a cofinal subset of α of ordertype at most ω, and such that for every club

$$C \subseteq \omega_1$$

there is a limit ordinal

$$\alpha < \omega_1$$

such that for all but finitely many $\beta \in b_\alpha$,

$$C \cap [\beta, \min(b_\alpha \setminus (\beta + 1))$$

is nonempty.

Club Guessing implies the Interval Hitting Principle.

Theorem 11. *Assuming the consistency of AD, there is a model of ZFC in which* $(+)^{\mathfrak{c}}_{<\omega}$ *and Club Weak Club Guessing hold and in which the Interval Hitting Principle fails.*