The Arithmetical Hierarchy in the Setting of ω_1 - Computability

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A.H. in ω_1 - computability

- Joint work with Jacob Carson, Julia Knight, Karen Lange, Charles McCoy, John Wallbaum.
- *The Arithmetical hierarchy in the setting of ω_1 - computability*, preprint.
- Continuation of work from N. Greenberg and J. F. Knight, *Computable structure theory in the setting of ω_1.*

Two definitions for the arithmetical hierarchy

We will give two definitions for the arithmetical hierarchy in the setting of ω_1 - computability.

- *The first will resemble the definition of the effective Borel Hierarchy.*
- *The second will resemble the standard definition of the hyper-arithmetical hierarchy.*

ω_1 - computability

Definition

Suppose R is a relation of countable arity α.

- R is **computably enumerable** if the set of ordinal codes for sequences in R is definable by a Σ_1 formula in (L_{ω_1}, ϵ).
- R is **computable** if it is both c.e. and co-c.e.

Working in ω_1

- We assume that $\mathbb{P}(\omega) \subseteq L_{\omega_1}$.
- Results of Gödel give a computable 1-1 function g from the countable ordinals onto L_{ω_1}, such that the relation $g(\alpha) \in g(\beta)$ is computable.
- So, computing in ω_1 is essentially the same as computing in L_{ω_1}.

Indices for c.e. sets

- As in the standard setting, we have a c.e. set of codes for Σ_1 definitions.
- We write W_α for the c.e. set with index α.
- All these definitions relativize in the natural way.

The jump

Definition

- We define the **halting set** as $K = \{\alpha : \alpha \in W_\alpha\}$.
- For a arbitrary set X, $X' = \{\alpha : \alpha \in W_\alpha^X\}$.
- $X^{(0)} = X$.
- $X^{(\alpha+1)} = (X^{(\alpha)})'$.
- For limit λ, $X^{(\lambda)}$ is the set of codes for pairs (β, x) such that $\beta < \lambda$ and $x \in X^{(\beta)}$.

- *We write Δ_n^0 for $\varnothing^{n-1}$ for $1 \leq n < \omega$.*
- *We write Δ_α^0 for $\varnothing^\alpha$ for $\alpha \geq \omega$.*

First definition for the arithmetical hierarchy

Our first definition of the arithmetical hierarchy resembles the definition of the effective Borel hierarchy.

Definition

Let R be a relation.

- R is Σ^0_0 and Π^0_0 if it is computable.
- R is Σ^0_1 if it is c.e.; R is Π^0_1 if the complementary relation, $\neg R$, is c.e.
- For countable $\alpha > 1$, R is Σ^0_α if it is **a c.e. union of relations, each of which is Π^0_β for some** $\beta < \alpha$; R is Π^0_α if $\neg R$ is Σ^0_α.

Indices for Σ^0_α and Π^0_α sets

For $\alpha \geq 1$, we may assign indices for the Σ^0_α and Π^0_α sets in the natural way.

- For $\alpha = 1$, we write $(\Sigma, 1, \gamma)$ as the index for the c.e. set with index γ.
- The set with index $(\Pi, 1, \gamma)$ is the complement.
- For $\alpha > 1$, the set with index (Σ, α, γ) is the **union of sets with indices in W_γ of the form** (Π, β, δ) for some $\beta < \alpha$ and some countable δ.
- The set with index (Π, α, γ) is the complement.

Second definition for the arithmetical hierarchy

Our second definition for the arithmetical hierarchy resembles the standard definition for the hyper-arithmetical hierarchy.

Definition

Let R be a relation.

- R is Σ^0_0 and Π^0_0 if it is computable.
- R is Σ^0_1 if it is c.e.; R is Π^0_1 if $\neg R$, is c.e.
- For $\alpha > 1$, R is Σ^0_α if it is **c.e. relative to** Δ^0_α; R is Π^0_α if $\neg R$ is Σ^0_α.

We assign indices for the Σ^0_α and Π^0_α sets in the same way.

Comparing the two definitions

The two definitions agree at finite levels, but disagree at level ω and beyond.

- Under the first definition, membership of an element into a Σ^0_α set occurs if and only if that element is a member of one of the lower Π^0_β sets.
- So membership into a Σ^0_α set uses information from a single lower level.
- Under the second definition, membership of an element into a Σ^0_α set may use a Δ^0_α oracle to get information from all lower levels simultaneously.

The two definitions disagree at level ω

Proposition

There is a set S that is Δ^0_ω under the second definition, but is not Σ^0_ω under the first definition.

Proof of the proposition

Proof.

- Define S such that $\alpha \in S$ iff α is not in the set with index (Σ, ω, α) under the first definition.
- For each n, α, let $S_{\alpha,n}$ be the union of the Σ^0_n sets with indices in W_α of the form (Π, k, β) with $k < n$.
- The union of these sets over all n will be the set with index (Σ, ω, α).
- A Δ^0_ω oracle can determine whether $\alpha \in S_{n,\alpha}$ for all n. So S is Δ^0_ω under the second definition.
- However, S cannot be one of the Σ^0_ω sets under the first defintion.

□

Computable infinitary formulas

The first definition of the computable infinitary formulas corresponds to the first definition of the arithmetical hierarchy.

Definition

Let L be a predicate language with computable symbols. We consider L-formulas $\varphi(\bar{x})$ with a countable tuple of variables $\bar{x}$.

- $\varphi(\bar{x})$ is **computable** Σ_0 and **computable** Π_0 if it is a quantifier-free formula of $L_{\omega_1,\omega}$.
- For $\alpha > 0$, $\varphi(\bar{x})$ is **computable** Σ_α if $\varphi \equiv \bigvee\!\!\!\!\bigvee_{c.e.} (\exists \bar{u}) \psi_i(\bar{u}, \bar{x})$, where each ψ_i is computable Π_β for some $\beta < \alpha$.
- $\varphi(\bar{x})$ is **computable** Π_α if $\varphi \equiv \bigwedge\!\!\!\!\bigwedge_{c.e.} (\forall \bar{u}) \psi_i(\bar{u}, \bar{x})$, where each ψ_i is computable Σ_β for some $\beta < \alpha$.

Computable infinitary formulas

The second definition of the computable infinitary formulas corresponds to the second definition of the arithmetical hierarchy.

Definition

- $\varphi(\bar{x})$ is **computable** Σ_0 and **computable** Π_0 if it is a quantifier-free formula of $L_{\omega_1,\omega}$.
- For $\alpha > 0$, $\varphi(\bar{x})$ is **computable** Σ_α if $\varphi \equiv \bigvee\!\!\!\bigvee_{c.e.} (\exists \bar{u}) \psi_i(\bar{u}, \bar{x})$, where each ψ_i is a **countable conjunction of formulas**, each computable Π_β for some $\beta < \alpha$.
- $\varphi(\bar{x})$ is **computable** Π_α if $\varphi \equiv \bigwedge\!\!\!\bigwedge_{c.e.} (\forall \bar{u}) \psi_i(\bar{u}, \bar{x})$, where each ψ_i is a **countable disjunction of formulas**, each computable Σ_β for some $\beta < \alpha$.

Proposition on computable infinitary formulas

Using either one of the definitions for the computable infinitary formulas, the following proposition holds and is proved by induction on α.

Proposition

Let $\mathcal{A}$ be an L-structure, and let $\varphi(\bar{x})$ be a computable Σ_α (computable Π_α) L-formula. Then the relation defined by $\varphi(\bar{x})$ in $\mathcal{A}$ is Σ^0_α (Π^0_α) relative to $\mathcal{A}$.

Relatively intrinsically arithmetical relations

Definition

- Let $\mathcal{A}$ be a computable structure, and let R be a relation on $\mathcal{A}$.
- We say that R is **relatively intrinsically** Σ^0_α on $\mathcal{A}$ if for all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is $\Sigma^0_\alpha(\mathcal{B})$.

Main theorem

We now present our main theorem.

Theorem

Let $1 \leq \alpha < \omega_1$. For a relation R on a computable structure $\mathcal{A}$, the following are equivalent:

1. *R is relatively intrinsically Σ^0_α on $\mathcal{A}$.*
2. *R is defined by a computable Σ_α formula.*

Idea of the proof

- The theorem requires two proofs, one for each definition of the arithmetical hierarchy.
- In either case, the proof for $2 \Rightarrow 1$ follows directly from the proposition.
- This is because a computable Σ_α formula is $\Sigma^0_\alpha(\mathcal{B})$ for any structure B. So it must be relatviely intrinsicaly Σ^0_α in $\mathcal{A}$.
- The proof for $1 \Rightarrow 2$ invokes the use of forcing by building an isomorphism from a generic copy $\mathcal{B}$ onto $\mathcal{A}$, where our forcing elements are partial isomorphisms.
- The proof is similar to that of the analogous result in the standard setting.

Which definition is better?

- It is not very efficacious to have two defintions for the arithmetical hierarchy.
- The authors believe that the second definition is a more natural definition.
- Consider our previous construction of the set that highlighted the differences in the defintions.
- In the standard setting, a element enters a Σ^0_5 set based on finitely much Δ^0_5 information.
- It seems natural that a membership into a Σ^0_ω set should use countably much Δ^0_ω information.

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