Coherence, NIP, UDTFS

Hunter Johnson

John Jay College, CUNY

a	b	c	d	
0	1	0	1	= p_1
0	0	0	1	= p_2
0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
1	0	0	1	= p_6

A finite φ -type space.

$$p_1 = \{ \neg\varphi(\bar{x}, a), \varphi(\bar{x}, b), \neg\varphi(\bar{x}, c), \varphi(\bar{x}, d) \}$$

Independence Dimension = 2

Independent Subsets of {a,b,c,d}:

{}

{a},{b},{c},{d}

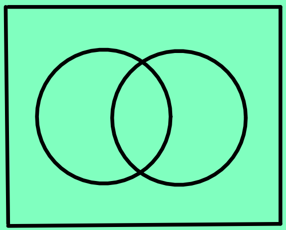
{a.b},{a,d}

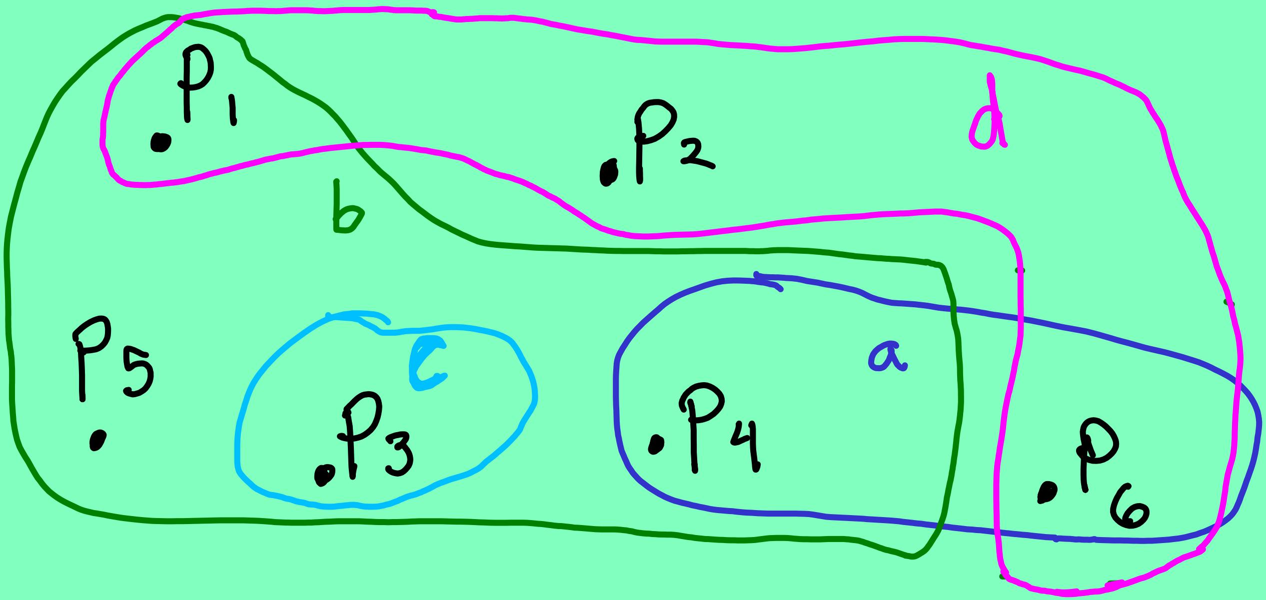
a	b	c	d	
0	1	0	1	= p_1
0	0	0	1	= p_2
0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
1	0	0	1	= p_6

Independent

{}
{a},{b},{c},{d}
{a.b},{a,d}

Independence





Definition: A formula $\varphi(\bar{x};\bar{y})$ is said to be NIP

(not the independence property) if $S_\varphi(M^{|\bar{y}|})$

has no infinite independent set.

The dual notion, Vapnik-Chervonenkis dimension posited independently

by Vapnik and Chervonkenkis about the same time.

a	b	c	d	
0	1	0	1	= p_1
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0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
1	0	0	1	= p_6

UDTFS

Independent

{}
{a},{b},{c},{d}
{a.b},{a,d}

$$\psi(y; z_1, z_2) = (y = z_1 \vee y = z_2)$$

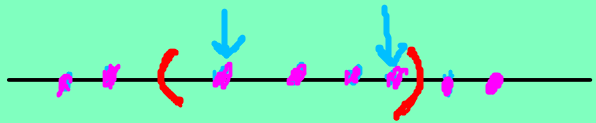
p_1 defined by $\psi(y, b, d)$

p_2 defined by $\psi(y, d, d)$

etc.

Some unstable UDTFS families

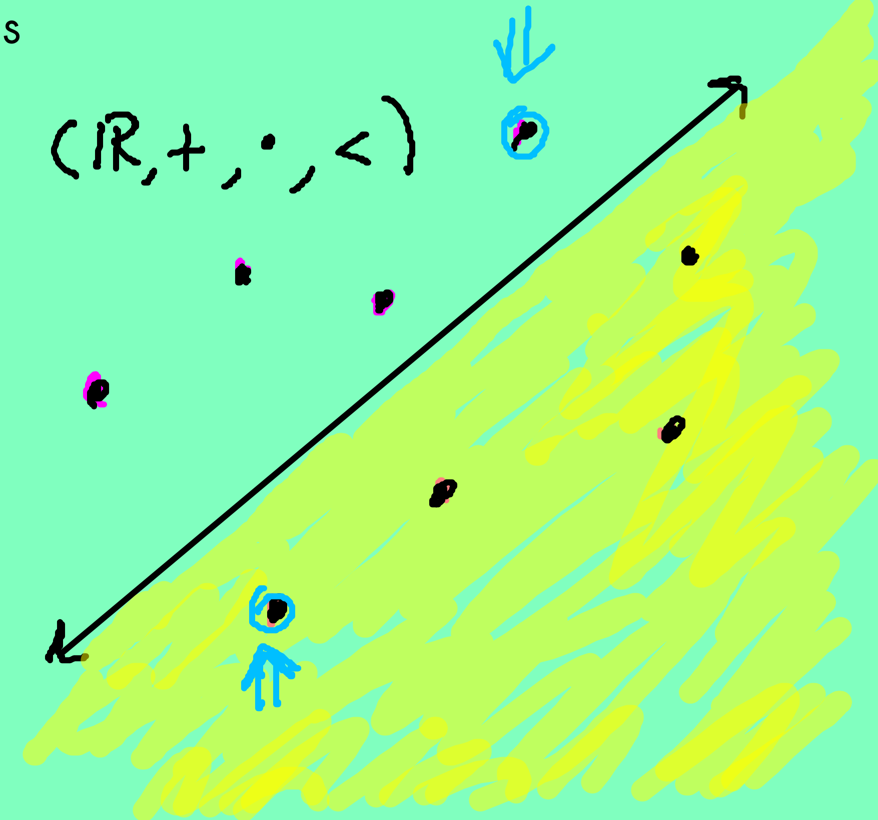
$(\mathbb{Q}, <)$



$\varphi(\bar{x}; \bar{y}) =$

$x_1 < y < x_2$

$(\mathbb{R}, +, \cdot, <)$



$y_1 \leq x_1 y_2 + x_2$

Conjecture: NIP <------> UDTFS (M. Warmuth, M.C. Laskowski)

Easy: UDTFS -----> NIP

Facts:

NIP + X --------> UDTFS

where X =
maximum (Floyd,Warmuth)
stable (Shelah)
(weakly) o-minimal (J., Laskowski)
VC minimal (Guingona)
dp-minimal (Guingona)

- Say that $\varphi(x,y)$ is *maximum* if ldim(φ) = d and for all finite B

$$\left| S_\varphi(B) \right| = \sum_{i=0}^{d} \binom{|B|}{i}$$

- Say that $\varphi(x,y)$ is *sub-maximum* if ldim(φ) = d and for all finite B

$$\left| S_\varphi(B) \right| < \sum_{i=0}^{d} \binom{|B|}{i}$$

Facts:

- If φ is sub-maximum of Idim 2, then φ is UDFTS (Guingona)

- If φ is maximum then φ is UDTFS (Floyd, Warmuth)

Question:

If φ is Idim = 2, then φ is UDTFS?

Not obvious.

Fact:

If φ is Idim 2, but not 2-maximum on an infinite subset of the monster, then φ is UDTFS

An interesting property of maximum φ :

Say that $p \in S_\varphi(A)$, A possibly infinite, has a root if there is some $A_0 \subseteq A$, independent and non-extendable in A, such that for any $a \in \mathfrak{C}^{|\bar{x}|}$,

$tp_\varphi(a/A_0)$ extends to p.

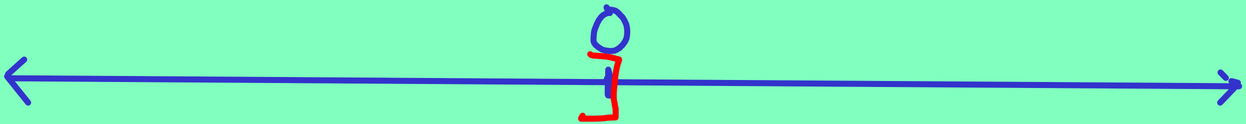
Example:

$$\varphi(x,y) := y \le x$$

Roots

$$A_0 = \{0\}$$

$(\mathbb{Q}, <)$



$$p = tp_\varphi(\langle 0 \rangle / \mathbb{Q})$$

$$p' = tp_\varphi(\langle 0-\delta \rangle / \mathbb{Q}),$$

$$0 < \delta < \mathbb{Q}^+$$

Fact:

If $p \in S_\varphi(A)$, and A_0 is a root of p,

then p is definable over A_0.

How?

If $a \in A \setminus A_0$ and $\varphi(x,a)^t \in p$

then t is the unique truth value of $\varphi(x,a)$ consistent with every trace on A_0. Otherwise $A_0 \cup \{a\}$ would be independent.

Fact: (Warmuth, Welzl)

If $\varphi(x,y)$ is maximum of Idim d, then for any finite B and $p \in S_\varphi(B)$, p has a root in B of size d.

Corollary: Maximum --> UDTFS

Question:

Can we fix this up so that it works with p that may not have roots of full size (Idim)?

If so, show NIP = UDTFS

a	b	c	d	
0	1	0	1	= p_1
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0	1	1	0	= p_3
1	1	0	0	= p_4
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1	0	0	1	= p_6

Independent

{}
{a},{b},{c},{d}
{a.b},{a,d}

- Guingona's quasi-order on bounded restrictions:

If $p \in S_\varphi(B)$, $\beta, \beta' \subseteq B$

say $\beta \leq_p \beta'$ if $p_\beta \vdash p_{\beta'}$.

Ex. $\{a,b\} \leq_{p_4} \{c,d\}$

a	b	c	d	
0	1	0	1	= p_1
0	0	0	1	= p_2
0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
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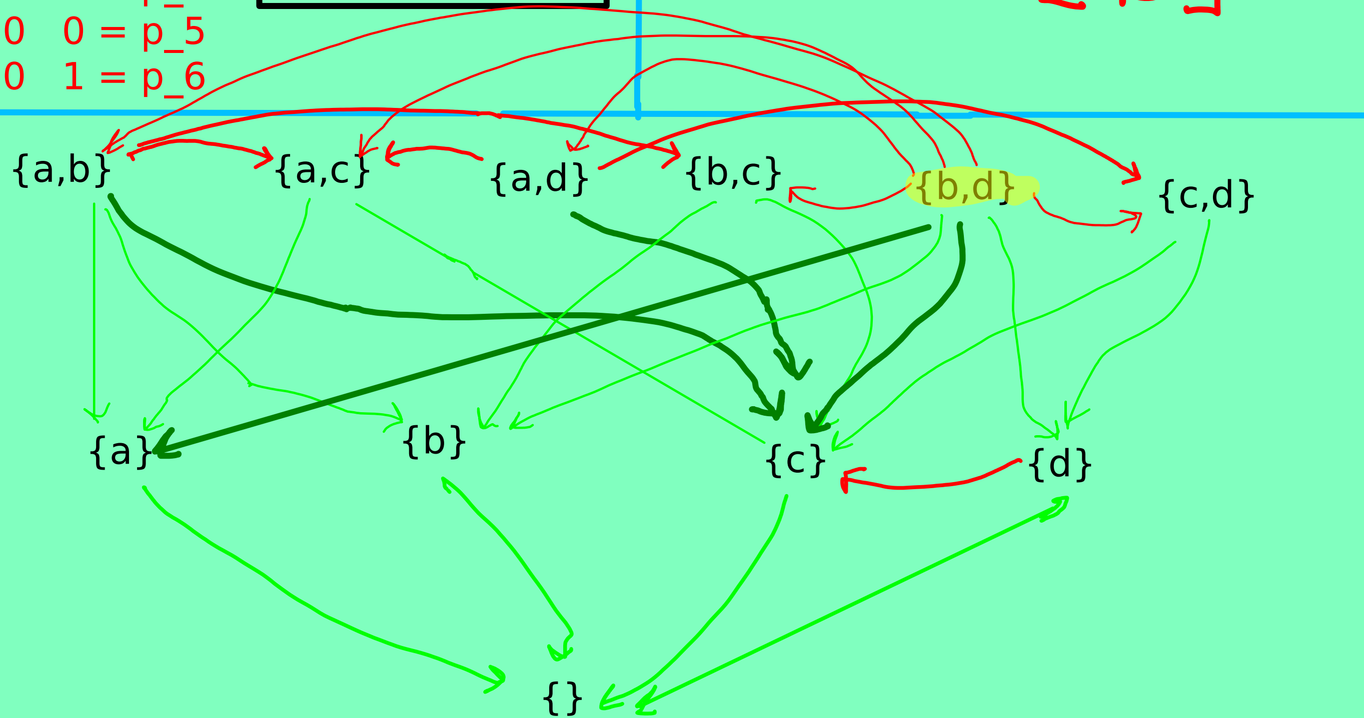
Independent

$$\boxed{\begin{array}{l} \{\} \\ \{a\},\{b\},\{c\},\{d\} \\ \{a.b\},\{a,d\} \end{array}}$$

Every $p \in S_\varphi(B)$ induces a quasi order on $[B]^{\leq k}$

$\leq_{p_1}$



- Problem:

  There may not be a uniform k so that all types are isolated by a size k subtype.

- Attempt at a solution:

  What if in addition to $\leq_p$ we are allowed to recruit the quasi-orders of "neighboring" types?

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0	1	0	1	= p_1
0	0	0	1	= p_2
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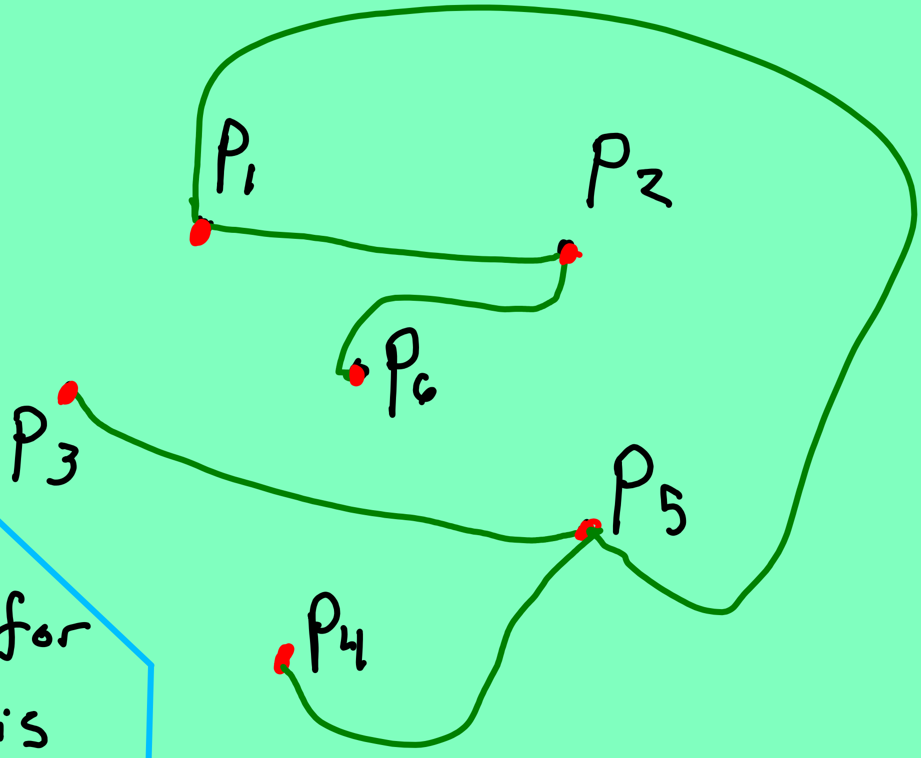
The 1-off graph



There are no strong elements in $[B]^{\le 2}$ for $\le_{p_2}$. But $\{b, d\}$ is strong if we are allowed $\le_{p_1}$.

Guingona's result on dp-minimal structures:

Given $\varphi(x, \bar{y})$ there is a natural number k such that

For any $p \in S_\varphi(B)$, with B finite, there is

$B' \in [B]^k$ such that:

If $b \in B$ is not in the part of p

isolated by $p \restriction_{B'}$, then there is a neighboring type

p' with $B' \leq_{p'} \{b\}$.

Moreover, for any such p',

$$\varphi(x,b) \in p' \iff \varphi(x,b) \in p$$

Coherence is an attempt to employ this idea outside of the dp-minimal context.

a	b	c	d	
0	1	0	1	= p_1
0	0	0	1	= p_2
0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
1	0	0	1	= p_6

Independent

{}
{a},{b},{c},{d}
{a.b},{a,d}

Basic Goal:

Given $p \in S_\varphi(B)$, try to find some independent set from which we can reconstruct p.

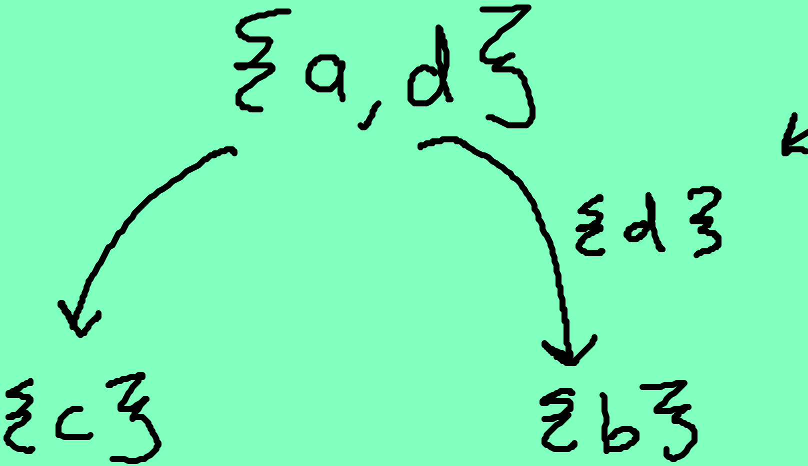
a	b	c	d	
0	1	0	1	= p_1
0	0	0	1	= p_2
0	1	1	0	= p_3
1	1	0	0	= p_4
0	1	0	0	= p_5
1	0	0	1	= p_6

Independent

{}
{a},{b},{c},{d}
{a.b},{a,d}

Consider p_1 and {a,d}

A perturbation



Fact:

If $B_0 \subseteq B$ is independent and non-extendable, then

for any $b \in B \setminus B_0$

there is a such that $tp_\varphi(a/B_0) \vdash tp_\varphi(a/B_0 b)$

Definition:

We say that a decides b over B_0 if

$$tp_\varphi(a/B_0) \vdash tp_\varphi(a/B_0 b)$$

Definition: Given $b \in B$ and $B_0 \subseteq B$, let

$$f(b, B_0) = \{a \in C : a \text{ decides } b \text{ over } B_0\}$$

Definition: $b, b' \in B$ are said to be inseparable over $B_0 \subseteq B$ if

$$f(b, B_0) = f(b', B_0)$$

and

$$f(b, B_0) \neq \emptyset.$$

Definition: $B_0 \subseteq B$ is coherent at $p \in S_\varphi(B)$

if whenever $b, b' \in B$ are inseparable over B_0,

$$\forall a \in f(b, B_0)$$

$$a \models p\restriction_{\{b\}} \wedge a \models p\restriction_{\{b'\}}$$

or $$a \not\models p\restriction_{\{b\}} \wedge a \not\models p\restriction_{\{b'\}}$$

Theorem:

Suppose that for some k, for all $p \in S_\varphi(B)$ with B finite, there exists $B_0 \in [B]^k$ such that B_0 is coherent at p. Then $\varphi(x, y)$ is UDTFS.

Proof:

The condition allows us to definably partition B into boundedly many partition elements, with p constant on each region.

Remarks

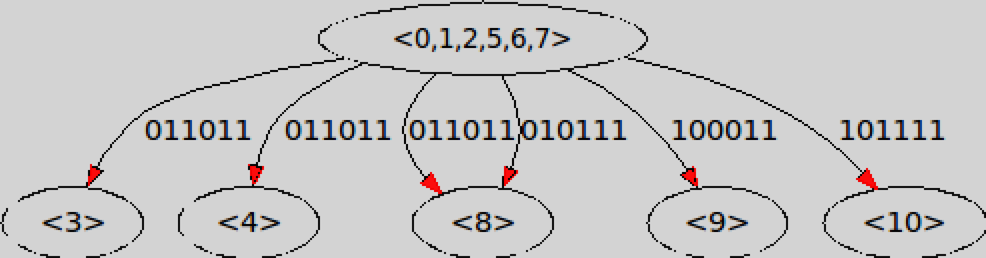
- The k in the above theorem will bound the combinatorial (or VC) density of the formula $\varphi(x, y)$.

- Guingona's argument in the dp-minimal case proves that each formula in a single x variable is coherent, a fortiori.

- Coherence makes sense for infinite sets B.
  Finiteness must be somehow used to prove the existence of coherence (eg by using the Guingona quasi-order) if the formula is strictly NIP.

Remarks:

- What is it good for? Condition is hard to prove.
- It does allow us to tell "at a glance" whether a given finite type space has compression of types down to size d=Idim.
- Conjecture: For any given p, some independent set is always coherent.



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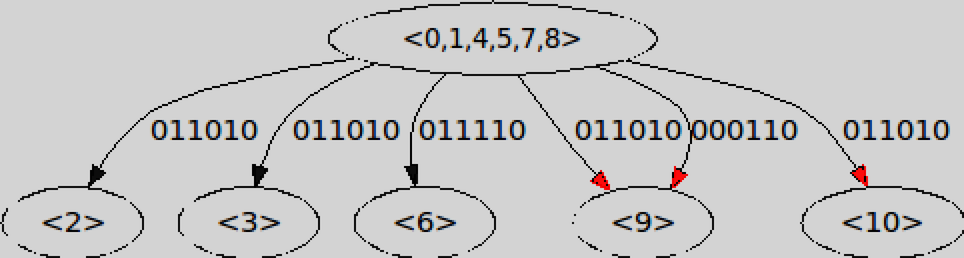
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