On Uniform Definability of Types over Finite Sets

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Outline

Types

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A φ-**type** p over B is a maximal collection of consistent formulas of the form $\pm\varphi(\overline{x};\overline{b})$ for various $\overline{b} \in B$.

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Definition.

The φ-**Stone Space** over B, denoted $S_\varphi(B)$, is the set of all φ-types over B.

Stability and Dependence

Definition.

We say a partitioned formula $\varphi(\overline{x};\overline{y})$ is **stable** if there do not exist $\langle \overline{a}_i : i < \omega \rangle$ and $\langle \overline{b}_j : j < \omega \rangle$ such that, for all $i, j < \omega$

$$\models \varphi(\overline{a}_i; \overline{b}_j) \text{ if and only if } i < j.$$

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Definition.

We say a partitioned formula $\varphi(\overline{x};\overline{y})$ is **dependent** (or sometimes **NIP**) if there do not exist $\langle \overline{a}_s : s \in \mathcal{P}(\omega)\rangle$ and $\langle \overline{b}_j : j < \omega\rangle$ such that, for all $s \in \mathcal{P}(\omega), j < \omega$

$$\models \varphi(\overline{a}_s; \overline{b}_j) \text{ if and only if } j \in s.$$

Definability of Types

A main property of stability, which we wish to generalize to dependence, is definability of types.

Definition.

Fix a formula $\varphi(\overline{x}; \overline{y})$, a φ-type p, and a parameter-definable formula $\psi(\overline{y})$. We say that ψ **defines** p if, for all $\overline{b} \in \mathrm{dom}(p)$, we have that

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Theorem (Shelah).

A partitioned formula $\varphi(\overline{x};\overline{y})$ is stable if and only if there exists formulas $\psi_k(\overline{y};\overline{z}_1,...,\overline{z}_n)$ for $k < K$ (finite) such that, for all non-empty sets $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_\varphi(B)$, there exists $\overline{c}_1,...,\overline{c}_n \in B$ and $k < K$ such that, $\psi_k(\overline{y};\overline{c}_1,...,\overline{c}_n)$ defines p.

Counting Type Spaces

Corollary.

If $\varphi(\overline{x};\overline{y})$ is stable, then there exists $K, n < \omega$ such that, for any non-empty set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

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Theorem (Sauer's Lemma).

If $\varphi(\overline{x}; \overline{y})$ is dependent, then there exists $K, n < \omega$ such that, for any non-empty *FINITE* set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

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Theorem (Sauer's Lemma).

If $\varphi(\overline{x};\overline{y})$ is dependent, then there exists $K, n < \omega$ such that, for any non-empty *FINITE* set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

Definition.

We say that a dependent formula φ has **VC-density** ℓ if ℓ is the infimum of all $n \in \mathbb{R}_+$ such that the condition in the above theorem holds.

Uniform Definability of Types over Finite Sets

Definition.

We say a partitioned formula $\varphi(\overline{x};\overline{y})$ has **UDTFS** if there exists formulas $\psi_k(\overline{y};\overline{z}_1,\ldots,\overline{z}_n)$ for $k < K$ such that, for all non-empty *FINITE* sets $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and all $p \in S_\varphi(B)$, there exists $\overline{c}_1,\ldots,\overline{c}_n \in B$ and $k < K$ such that $\psi_k(\overline{y};\overline{c}_1,\ldots,\overline{c}_n)$ defines p. A theory T has UDTFS if all partitioned formulas do.

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Definition.

We will say that a formula φ with UDTFS has **UDTFS rank** n if n is minimal such.

Facts about UDTFS

Facts.

1. If $\varphi(\overline{x};\overline{y})$ is stable, then φ has UDTFS.
2. If $\varphi(\overline{x};\overline{y})$ has UDTFS rank n, then the VC-density of φ is $\leq n$.
3. If $\varphi(\overline{x};\overline{y})$ has UDTFS, then φ is dependent.

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Theorem (Johnson, Laskowski).

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Theorem (Johnson, Laskowski).

If T is o-minimal, then T has UDTFS.

$$\begin{array}{ccccc} & & \text{stable} & & \\ & & \Downarrow & & \\ \text{o-minimal} & \Rightarrow & \text{UDTFS} & \Rightarrow & \text{dependent} \end{array}$$

dp-Minimal Theories

Definition.

A theory T is **dp-minimal** if there do not exist $\varphi(x;\overline{y})$, $\psi(x;\overline{z})$, $\langle \overline{b}_i : i < \omega \rangle$, and $\langle \overline{c}_j : j < \omega \rangle$ such that, for all $i_0, j_0 < \omega$, the type

$$\{\neg\varphi(x;\overline{b}_{i_0}), \neg\psi(x;\overline{c}_{j_0})\} \cup \{\varphi(x;\overline{b}_i) : i \neq i_0\} \cup \{\psi(x;\overline{c}_j) : j \neq j_0\}.$$

is consistent.

Examples of dp-Minimal Theories

Examples.

The following theories are dp-minimal:

1. Any o-minimal theory or weakly o-minimal theory,
2. $\mathrm{Th}(\mathbb{Z}; <, +)$,
3. $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |, 0, 1)$ (where $x|y$ iff. $v_p(x) \leq v_p(y)$),
4. Algebraically closed valued fields.
5. In general, any VC-minimal theory is dp-minimal.
6. Any theory with VC-density ≤ 1 is dp-minimal.

dp-Minimal Theories have UDTFS

Theorem (G.).

If T is dp-minimal, then T has UDTFS.

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If $\varphi(\overline{x};\overline{y})$ and $N < \omega$ are such that, for all $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| = N$, $|S_\varphi(B)| \leq N(N+1)/2$, then φ has UDTFS (in particular if φ has VC-density < 2, then φ has UDTFS).

Valued Fields and UDTFS

Theorem (G.).

If (K, k, Γ) is a Henselian valued field that has elimination of field quantifiers in the Denef-Pas language, $\mathrm{Th}(k)$ has UDTFS, and $\mathrm{Th}(\Gamma)$ has UDTFS, then the full theory in the Denef-Pas language has UDTFS.

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Examples.

The theories of the following structures in the Denef-Pas language have UDTFS:

1. $\mathbb{Q}_p$,
2. $\mathbb{R}((t))$,
3. $\mathbb{C}((t))$,
4. $\mathbb{C}((t^{\mathbb{Q}}))$.

Maximum Formulas have UDTFS

Definition.

A partitioned formula $\varphi(\overline{x};\overline{y})$ is **maximum of dimension** d if, for all finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$,

$$\left|S_{\varphi}(B)\right| = \sum_{i \leq d} \binom{|B|}{i}.$$

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The following proposition follows from the work of Floyd and Warmuth.

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The following proposition follows from the work of Floyd and Warmuth.

Proposition.

If φ is maximum of dimension d, then φ has UDTFS. Furthermore, it has UDTFS rank $\leq d$.

The UDTFS Conjecture

$$\begin{array}{ccccccccc}
\text{o-minimal} & \Rightarrow & \text{VC-density} \leq 1 & & \text{stable} & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{VC-minimal} & \Rightarrow & \text{dp-minimal} & \Rightarrow & \text{UDTFS} & \Rightarrow & \text{dependent}
\end{array}$$

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Open Question (Laskowski).

If φ is dependent, then does φ have UDTFS?

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More Open Questions.

1. Is UDTFS closed under reducts?

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More Open Questions.

1. Is UDTFS closed under reducts?
2. If $\varphi(\overline{x};\overline{y})$ has UDTFS, then does $\varphi^{\text{opp}}(\overline{y};\overline{x})$ have UDTFS?

Rank Relations

Recall.

The following hold for any partitioned formula $\varphi(\overline{x};\overline{y})$:

1. φ is dependent if and only if φ has finite VC-density.
2. The VC-density of φ is bounded by the UDTFS rank of φ.

Sufficiency of a Single Variable

Proposition (G.).

If T is such that all formulas of the form $\varphi(x; \overline{y})$ have UDTFS rank $\leq k$, then all formulas of the form $\varphi(\overline{x}; \overline{y})$ have UDTFS rank $\leq k \cdot \lg(\overline{x})$.

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Corollary.

If T is such that all formulas of the form $\varphi(x;\overline{y})$ have UDTFS rank $\leq k$, then T has VC-density $\leq k$.

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Corollary.

If T is such that all formulas of the form $\varphi(x;\overline{y})$ have UDTFS rank $\leq k$, then T has VC-density $\leq k$.

The following is originally due to Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko, but follows as a corollary of the above proposition:

Corollary.

If T is weakly o-minimal, then T has VC-density ≤ 1.

Future Work: Kueker Conjecture

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The Kueker Conjecture.

If T is a theory in a countable language such that every uncountable model of T is $\aleph_0$-saturated, then T is $\aleph_0$-categorical or $\aleph_1$-categorical.

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Theorem (Hrushovski).

1. If T is stable, then T satisfies the Kueker Conjecture.
2. If T interprets an infinite linear order, then T satisfies the Kueker Conjecture.

Partial Results for the Kueker Conjecture

Examples.

The following theories are VC-minimal:

1. Any o-minimal theory, including $\mathrm{Th}(\mathbb{R}; <, +, \cdot, 0, 1)$.
2. Any strongly minimal theory, including $\mathrm{Th}(\mathbb{C}; +, \cdot, 0, 1)$.
3. The theory of algebraically closed valued fields.

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Theorem (G.).

If T is VC-minimal, then T satisfies the Kueker Conjecture.

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