Definable operators on Hilbert spaces

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ASL North American Annual Meeting
Special Session in Model Theory
UC Berkeley
March 27, 2011

Continuous Logic

The Main Theorem

Corollaries

Continuous logic in a nutshell

- Metric structures are bounded complete metric spaces together with distinguished constants, functions, and predicates; however, predicates P now take values in closed, bounded intervals $I_P \subseteq \mathbb{R}$ rather than $\{0, 1\}$.
- The distinguished functions and predicates must also be uniformly continuous.
- Metric signatures provide symbols for these distinguished constants, functions and predicates. Moreover, they specify the intervals I_P as well as a modulus of uniform continuity for which their interpretations must obey.
- For the moment, let's assume that $I_P = [0, 1]$ for all predicates P and let us assume that $d \leq 1$.

Continuous logic in a nutshell (cont'd)

- Atomic formulae are now of the form $d(t_1, t_2)$ and $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and P is a predicate symbol.
- We allow all continuous functions $[0,1]^n \to [0,1]$ as n-ary connectives.
- $\forall x$ and $\exists x$ are replaced by $\sup_x$ and $\inf_x$.

Definable predicates

- If M is a metric structure and $\varphi(x)$ is a formula, where $|x| = n$, then the interpretation of φ in M is a uniformly continuous function $\varphi^M : M^n \to [0, 1]$.
- For the purposes of definability, formulae are not expressive enough. Instead, we broaden our perspective to include *definable predicates*.
- If $A \subseteq M$, then a uniformly continuous function $P : M^n \to [0, 1]$ is *definable in M over A* if there is a sequence $(\varphi_n(x))$ of formulae with parameters from A such that the sequence (φ_n^M) converges uniformly to P.

Definable sets and functions

- $X \subseteq M^n$ is *A-definable* if and only if X is closed and the map $x \mapsto d(x, X) : M^n \to [0, 1]$ is an A-definable predicate.
- $f : M^n \to M$ is *A-definable* if and only if the map $(x, y) \mapsto d(f(x), y) : M^{n+1} \to [0, 1]$ is an A-definable predicate.
- A new complication: Definable sets and functions may now use *countably* many parameters in their definitions. If the metric structure is separable and the parameterset used in the definition is dense, then this can prove to be troublesome.

Definability takes a backseat

- There are notions of stability, simplicity, rosiness, NIP,... in the metric context. These notions have been heavily developed with an eye towards applications.
- However, old-school model theory in the form of definability has not really been pursued. In particular, the question: "Given a metric structure M, what are the sets and functions definable in M?" has not received much attention. The following result appears to be the first result in this direction:

Theorem (G.-2010)

If $\mathfrak{U}$ denotes the Urysohn sphere and $f : \mathfrak{U}^n \to \mathfrak{U}$ is definable, then either f is a projection function or has relatively compact image.

Hilbert spaces

- Throughout, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- Recall that an inner product space over $\mathbb{K}$ which is complete with respect to the metric induced by its inner product is called a $\mathbb{K}$-Hilbert space. In this talk, H and H' denote *infinite-dimensional* $\mathbb{K}$-Hilbert spaces.
- A continuous linear transformation $T : H \to H'$ is also called a *bounded* linear transformation. Reason: if one defines

$$\|T\| := \sup\{\|T(x)\| \ : \ \|x\| \leq 1\},$$

then T is continuous if and only if $\|T\| < \infty$.
- We let $\mathfrak{B}(H)$ denote the (C^*-) algebra of bounded operators on H.

Signature for Real Hilbert Spaces

We work with the following many-sorted metric signature:

- for each $n \geq 1$, we have a sort for $B_n(H) := \{x \in H \mid \|x\| \leq n\}$.
- for each $1 \leq m \leq n$, we have a function symbol $I_{m,n} : B_m(H) \to B_n(H)$ for the inclusion mapping.
- function symbols $+, - : B_n(H) \times B_n(H) \to B_{2n}(H)$;
- function symbols $r\cdot : B_n(H) \to B_{kn}(H)$ for all $r \in \mathbb{R}$, where k is the unique natural number satisfying $k - 1 \leq |r| < k$;
- a predicate symbol $\langle \cdot, \cdot \rangle : B_n(H) \times B_n(H) \to [-n^2, n^2]$;
- a predicate symbol $\| \cdot \| : B_n(H) \to [0, n]$.

The moduli of uniform continuity are the natural ones.

Signature for Complex Hilbert Spaces

When working with complex Hilbert spaces, we make the following changes:

- We add function symbols $i\cdot : B_n(H) \to B_n(H)$ for each $n \geq 1$, meant to be interpreted as multiplication by i.
- Instead of the function symbol $\langle \cdot, \cdot \rangle : B_n(H) \times B_n(H) \to [-n^2, n^2]$, we have two function symbols $\mathfrak{Re}, \mathfrak{Im} : B_n(H) \times B_n(H) \to [-n^2, n^2]$, meant to be interpreted as the real and imaginary parts of $\langle \cdot, \cdot \rangle$.

Definable functions

Definition

Let $A \subseteq H$. We say that a function $f : H \to H$ is *A-definable* if:

(i) for each $n \geq 1$, $f(B_n(H))$ is bounded; in this case, we let $m(n,f) \in \mathbb{N}$ be the minimal m such that $f(B_n(H))$ is contained in $B_m(H)$;

(ii) for each $n \geq 1$ and each $m \geq m(n,f)$, the function

$$f_{n,m} : B_n(H) \to B_m(H), \quad f_{n,m}(x) = f(x)$$

is A-definable, that is, the predicate $P_{n,m} : B_n(H) \times B_m(H) \to [0,m]$ defined by $P_{n,m}(x,y) = d(f(x),y)$ is A-definable.

Lemma

The definable bounded operators on H form a subalgebra of $\mathfrak{B}(H)$.

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Statement of the Main Theorem

From now on, $I : H \to H$ denotes the identity operator.

Definition

An operator $K : H \to H$ is *compact* if $K(B_1(H))$ has compact closure. (In terms of nonstandard analysis: K is compact if and only if for all finite vectors $x \in H^*$, we have $K(x)$ is nearstandard.)

Theorem (G.-2010)

The bounded operator $T : H \to H$ is definable if and only if there is $\lambda \in \mathbb{K}$ and a compact operator $K : H \to H$ such that $T = \lambda I + K$.

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Finite-Rank Operators

- Suppose first that T is a *finite-rank* operator, that is, $T(H)$ is finite-dimensional.
- Let $a_1, \ldots, a_n$ be an orthonormal basis for $T(H)$. Then $T(x) = T_1(x)a_1 + \cdots + T_n(x)a_n$ for some bounded linear functionals $T_1, \ldots, T_n : H \to \mathbb{R}$.
- By the Riesz Representation Theorem, there are $b_1, \ldots, b_n \in H$ such that $T_i(x) = \langle x, b_i \rangle$ for all $x \in H$, $i = 1, \ldots, n$.
- Then, for all $x, y \in H$, we have

$$d(T(x), y) = \sqrt{\sum_{i=1}^{n}(\langle x, b_i \rangle^2) - 2\sum_{i=1}^{n}(\langle x, b_i \rangle \langle a_i, y \rangle) + \|y\|^2}$$

which is a formula in our language. Hence, finite-rank operators are strongly definable.

Compact Operators

Fact

If $T : H \to H$ is compact, then there is a sequence (T_n) of finite-rank operators such that $\|T - T_n\| \to 0$ as $n \to \infty$.

- Now suppose that $T : H \to H$ is a compact operator. Fix a sequence (T_n) of finite-rank operators such that $\|T - T_n\| \to 0$.
- Fix $n \geq 1$ and $\epsilon > 0$ and choose k such that $\|T - T_k\| < \frac{\epsilon}{n}$. Then for $x \in B_n(H)$ and $y \in B_m(H)$, where $m \geq m(n, T)$, we have

$$|d(T(x), y) - d(T_k(x), y)| \leq \|T(x) - T_k(x)\| < \epsilon.$$

- Since $d(T_k(x), y)$ is given by a formula, this shows that T is definable.
- Thus, any operator of the form $\lambda I + T$ is definable.

Working towards the converse

- From now on, we fix an A-definable operator $T : H \to H$, where $A \subseteq H$ is countable.
- We also let H^* denote an ω_1-saturated elementary extension of H.
- Observe that, since H is closed in H^*, we have the orthogonal decomposition $H^* = H \oplus H^\perp$.
- T has a natural extension to a definable function $T : H^* \to H^*$.

Lemma

$T : H^* \to H^*$ is also linear.

Proof.

Not as straightforward as you might guess given that continuous logic is a positive logic! □

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Definable closure

Facts

- In an arbitrary metric structure M, if $f : M \to M$ is an A-definable function, then $f(x) \in \mathrm{dcl}(Ax)$ for all $x \in M$.
- In a Hilbert space H, $\mathrm{dcl}(B) = \overline{\mathrm{sp}}(B)$, the closed linear span of B, for any $B \subseteq H$.

We let $P : H^* \to H^*$ denote the orthogonal projection onto the subspace $\overline{\mathrm{sp}}(A)$.

Lemma

For any $x \in H^*$, $\mathrm{dcl}(Ax) = \overline{\mathrm{sp}}(Ax) = \overline{\mathrm{sp}}(A) \oplus \mathbb{K} \cdot (x - Px)$.

Main Lemma

Lemma

There is a unique $\lambda \in \mathbb{K}$ *such that, for all* $x \in H^*$*, we have* $T(x) = PT(x) + \lambda(x - Px)$*.*

Proof.

- If $x \in H^\perp$, then there is $\lambda_x \in \mathbb{K}$ such that $T(x) = PT(x) + \lambda_x \cdot x$.
- It is easy to check that $\lambda_x = \lambda_y$ for all $x, y \in H^\perp$; call this common value λ.
- For $x \in H^*$ arbitrary, we have

$$T(x) = TP(x) + T(x - Px) = TP(x) + PT(x - Px) + \lambda(x - Px).$$

- Since $TP(x) + PT(x - Px) \in \overline{\text{sp}}(A)$, we are done.

□

Finishing the converse

Proposition

For λ as above, we have $T - \lambda I$ is a compact operator.

Proof

- Since $T - \lambda I = P \circ (T - \lambda I)$, we have $(T - \lambda I)(H^*) \subseteq \overline{\mathrm{sp}}(A)$.
- Let $\epsilon > 0$ be given. Let $\varphi(x, y)$ be a formula such that $\big|\|T(x) - y\| - \varphi(x, y)\big| < \frac{\epsilon}{4}$, where x is a variable of sort B_1.
- Let (b_n) be a countable dense subset of $(T - \lambda I)(B_1(H^*))$.
- Then the following set of statements is inconsistent:

$$\{\|T(x) - (\lambda x + b_n)\| \geq \frac{\epsilon}{4} \mid n \in \mathbb{N}\}.$$

Proof (cont'd)

- Thus, the following set of conditions is inconsistent:

$$\{\varphi(x, \lambda x + b_n) \geq \frac{\epsilon}{2} \mid n \in \mathbb{N}\}.$$

- By ω_1-saturation, there are $b_1, \ldots, b_m$ such that

$$\{\varphi(x, \lambda x + b_n) \geq \frac{\epsilon}{2} \mid 1 \leq n \leq m\}$$

  is inconsistent.
- It follows that $\{b_1, \ldots, b_m\}$ form an ϵ-net for $(T - \lambda I)(B_1(H^*))$.
- Since $\epsilon > 0$ is arbitrary, we see that $(T - \lambda I)(B_1(H^*))$ is totally bounded. It is automatically closed by ω_1-saturation, whence it is compact. □

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Corollaries

Some Corollaries- I

Corollary

The definable operators on H form a C^*-subalgebra of $\mathfrak{B}(H)$.

- It is not at all clear how to prove, from first principles, that definable operators are closed under taking adjoints.
- It is easy to show this if one assumes that the definable operator is *normal*, for then one has

$$\begin{aligned}\|T^*(x)-y\|^2 &= \|T^*(x)\|^2 - 2\langle T^*(x), y\rangle + \|y\|^2 \\ &= \|T(x)\|^2 - 2\langle T(y), x\rangle + \|y\|^2.\end{aligned}$$

Some Corollaries-II

Corollary

Suppose that T is definable and not compact. Then $\mathrm{Ker}(T)$ *and* $\mathrm{Coker}(T)$ *are finite-dimensional. Moreover,* $\mathrm{Ker}(T) \subseteq \overline{\mathrm{sp}}(A)$.

Proof.

- The moreover is clear from the main lemma.
- By taking adjoints, it is enough to prove the result for $\mathrm{Ker}(T)$.
- Let $\varphi_k(x, y)$ approximate $d(T(x), y)$ within an error of $\frac{1}{k}$. Then the following set of formulae is inconsistent:

$$\{\varphi_k(x,0) \leq \frac{1}{k} \ : \ k \geq 1\} \cup \{d(x,a) \geq \epsilon \mid a \in A\}$$

- By ω_1-saturation, there is a finite ϵ-net for $B_1(\mathrm{Ker}(T))$. Thus, $B_1(\mathrm{Ker}(T))$ is compact, whence $\mathrm{Ker}(T)$ is finite-dimensional.

Some Corollaries- III

Corollary

Suppose that E is a closed subspace of H and that $T : H \to H$ is the orthogonal projection onto E. Then T is definable if and only if E has finite dimension or finite codimension.

Corollary

Let $I = \{i_1, i_2, \ldots\}$ be an infinite and coinfinite subset of $\mathbb{N}$. Let $T : \ell^2 \to \ell^2$ be given by $T(x)_n = x_{i_n}$. Then T is not definable.

Fredholm operators

From now on, we assume that $\mathbb{K} = \mathbb{C}$. Recall that a bounded operator T is *Fredholm* if both $\mathrm{Ker}(T)$ and $\mathrm{Coker}(T)$ are finite-dimensional. The *index* of a Fredholm operator is the number $\mathrm{index}(T) := \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Coker}(T))$.

Corollary

If T is definable, then either T is compact or else T is Fredholm of index 0.

Proof.

This follows from the Fredholm alternative of functional analysis. □

Some Corollaries- IV

Recall the left- and right-shift operators L and R on ℓ^2:

$$L(x_1, x_2, \ldots,) = (x_2, x_3, \ldots)$$

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots,)$$

Corollary

The left- and right-shift operators on ℓ^2 are not definable.

Proof.

These operators are of index 1 and -1 respectively. □

Using this result, one can prove that the left-and right-shift operators on the $\mathbb{R}$-Hilbert space ℓ^2 are not definable.

The Calkin Algebra

- Let $\mathfrak{B}_0(H)$ denote the ideal of $\mathfrak{B}(H)$ consisting of the compact operators. The quotient algebra $\mathfrak{C}(H) = \mathfrak{B}(H)/\mathfrak{B}_0(H)$ is referred to as the *Calkin algebra* of H.
- Let $\pi : \mathfrak{B}(H) \to \mathfrak{C}(H)$ be the canonical quotient map.
- Our main theorem says that the algebra of definable operators is equal to $\pi^{-1}(\mathbb{C})$.
- We consider the *essential spectrum* of T:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \ : \ \pi(T) - \lambda \cdot \pi(I) \text{ is not invertible}\}.$$

Some Corollaries- V

If T is a definable operator, let $\lambda(T) \in \mathbb{C}$ be such that $T - \lambda(T)I = P \circ (T - \lambda(T)I)$.

Corollary

If T is definable, then $\sigma_e(T) = \{\lambda(T)\}$.

Example

Consider $L \oplus R : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$.

- It is a fact that $L \oplus R$ is Fredholm of index 0. Thus, our earlier corollary doesn't help us in showing that $L \oplus R$ is not definable.
- However, it is a fact that $\sigma_e(L \oplus R) = \mathbb{S}^1$. Thus, we see from the above corollary that $L \oplus R$ is not definable.

The Invariant Subspace Problem

Invariant Subspace Problem

If H is a separable Hilbert space and $T : H \to H$ is a bounded operator, does there exist a closed subspace E of H such that $E \neq \{0\}$, $E \neq H$, and $T(E) \subseteq E$?

Silly Corollary

The invariant subspace problem has a positive answer when restricted to the class of *definable* operators.

Proof.

Suppose T is definable. Write $T = \lambda I + K$. If $K = 0$, then $E := \mathbb{C} \cdot x$ is a closed, nontrivial invariant subspace for T, where $x \in H \setminus \{0\}$ is arbitrary. Otherwise, use the fact that compact operators always have nontrivial invariant subspaces. □

Open Questions

Question 1

Can we characterize other definable functions in Hilbert spaces? What about nonlinear isometries?

Question 2

Are all definable functions on a Hilbert space “piecewise linear”?

Question 3

Can we characterize the definable operators in certain expansions of Hilbert spaces? E.g. Hilbert spaces equipped with a generic automorphism?

References

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Preprints for both papers are available at

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