I will describe a fascinating mathematical object, the field $\mathbb{T}$ of transseries. It is an ordered differential field extension of $\mathbb{R}$ and is a kind of universal domain for real differential algebra.

**Conjecture:** the elementary theory of $\mathbb{T}$ is model complete, and is the model companion of the theory of $H$-fields.

After discussing $\mathbb{T}$ we introduce $H$-fields, and then sketch some partial results towards this conjecture.

(Joint work with Aschenbrenner and van der Hoeven)
Reminder on Laurent series

The ordered differential field $\mathbb{R}((x^{-1}))$ of formal Laurent series in *descending* powers of $x$ over $\mathbb{R}$ consists of all series of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 + a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots$$

$x > \mathbb{R}$ for the ordering, $x' = 1$ for the derivation. **Defects:**

- $x^{-1}$ has no antiderivative $\log x$ in $\mathbb{R}((x^{-1}))$.
- There is no natural exponentiation defined on all of $\mathbb{R}((x^{-1}))$; such an operation should satisfy $\exp x > x^n$ for all $n$.

Exponentiation does make sense for the *finite* elements of $\mathbb{R}((x^{-1}))$:

$$\exp(a_0 + a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots)$$

$$= e^{a_0} \sum_{n=0}^{\infty} \frac{1}{n!} (a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots)^n$$

$$= e^{a_0} (1 + b_1 x^{-1} + b_2 x^{-2} + \cdots)$$
The field of transseries

To remove these defects we extend $\mathbb{R}((x^{-1}))$ to an ordered differential field $\mathbb{T}$ of transseries: series of transmonomials (or logarithmic-exponential monomials) arranged from left to right in decreasing order and multiplied by real coefficients, for example

$$e^{e^x} - 3e^{x^2} + 5x^{1/2} - \log x + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x}.$$

The reversed order type of the set of transmonomials that occur in a given transseries series can be any countable ordinal. (For the series displayed it is $\omega + 2$.) Such series occur for example in solving implicit equations of the form $P(x, y, e^x, e^y) = 0$ for $y$ as $x \rightarrow +\infty$, where $P$ is a polynomial in 4 variables over $\mathbb{R}$. The Stirling expansion for the Gamma function is also a transseries. Transseries also arise naturally as formal solutions to algebraic differential equations.
Transseries

Some typical computations in $\mathbb{T}$:

- **Taking a reciprocal**

\[
\frac{1}{x - x^2e^{-x}} = \frac{1}{x(1 - xe^{-x})} = x^{-1}(1 + xe^{-x} + x^2e^{-2x} + \cdots)
\]

\[
= x^{-1} + e^{-x} + xe^{-2x} + \cdots
\]

- **Formal Integration**

\[
\int \frac{e^x}{x} \, dx = constant + \sum_{n=0}^{\infty} n!x^{-1-n}e^x \quad (\text{diverges}).
\]

- **Formal Composition**

Let $f(x) = x + \log x$ and $g(x) = x \log x$. Then

\[
f(g(x)) = x \log x + \log(x \log x)
\]

\[
= x \log x + \log x + \log(\log x)
\]
Transseries

- **Formal Composition continued**

\[ g(f(x)) = (x + \log x) \log(x + \log x) \]

\[ = x \log x + (\log x)^2 + (x + \log x) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\log x}{x} \right)^n \]

\[ = x \log x + (\log x)^2 + \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n + 1)} \left( \frac{\log x}{x^n} \right)^{n+1}. \]

- **Compositional Inversion**

The transseries \( g(x) = x \log x \) has a compositional inverse of the form

\[ \frac{x}{\log x} \left( 1 + F\left( \frac{\log \log x}{\log x}, \frac{1}{\log x} \right) \right) \]

where \( F(X, Y) \) is an ordinary convergent power series in the two variables \( X \) and \( Y \) over \( \mathbb{R} \).
Some key properties of $\mathbb{T}$: it is a real closed ordered field extension of $\mathbb{R}$, and is equipped with natural operations of *exponentiation* ($\exp$) and (termwise) differentiation, $f \mapsto f'$, such that

\[
\exp(\mathbb{T}) = \mathbb{T}^>0, \quad \{f' : f \in \mathbb{T}\} = \mathbb{T}, \quad \{f \in \mathbb{T} : f' = 0\} = \mathbb{R}.
\]

As an exponential ordered field $\mathbb{T}$ is an elementary extension of the real exponential field. The iterated exponentials

\[
x, \exp x, \exp(\exp(x)), \ldots
\]

are cofinal in the ordering of $\mathbb{T}$.  


Conjectures about $\mathbb{T}$

From now on we consider $\mathbb{T}$ as an ordered differential field.

*Conjecture 1:* $\mathbb{T}$ is model complete.

*Conjecture 2:* If $X \subseteq \mathbb{T}^n$ is definable, then $X \cap \mathbb{R}^n$ is semialgebraic.

*Conjecture 3:* $\mathbb{T}$ is asymptotically o-minimal, that is, for each definable $X \subseteq \mathbb{T}$ either all sufficiently large $f \in \mathbb{T}$ are in $X$, or all sufficiently large $f \in \mathbb{T}$ are outside $X$. 
Asymptotic $o$-minimality holds for quantifier-free definable $X \subseteq T$.

Best evidence for model-completeness of $T$: the detailed analysis by van der Hoeven in “Transseries and Real Differential Algebra” (Springer Lecture Notes 1888) of the set of zeros in $T$ of any given differential polynomial in one variable over $T$. He proved:

**Theorem**

*Given any differential polynomial $P(Y) \in T\{Y\}$ and $f, h \in T$ with $P(f) < 0 < P(h)$, there is $g \in T$ with $f < g < h$ and $P(g) = 0$.***

Here and later $K\{Y\} = K[Y, Y', Y'', \ldots]$ is the ring of differential polynomials in the indeterminate $Y$ over a differential field $K$. 
Linear differential operators over $\mathbb{T}$

Another analogy with the real field is that linear differential operators over $\mathbb{T}$ behave much like one-variable polynomials over $\mathbb{R}$. A linear differential operator over $\mathbb{T}$ is an operator

$$A = a_0 + a_1 D + \cdots + a_n D^n$$

on $\mathbb{T}$ ($D$ = the derivation, all $a_i \in \mathbb{T}$); it defines the same function on $\mathbb{T}$ as the differential polynomial $a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)}$. The linear differential operators over $\mathbb{T}$ form a noncommutative ring under composition.

**Theorem**

*Each linear differential operator over $\mathbb{T}$ of order $n > 0$ is surjective as a map $\mathbb{T} \to \mathbb{T}$, and is a product (composition) of operators $a + bD$ of order 1 and operators $a + bD + cD^2$ of order 2.*
The role of $H$-fields

Abraham Robinson taught us to think about model completeness in an algebraic way. Accordingly, we introduce a class of ordered differential fields, the so-called $H$-fields. These are defined so as to share certain basic (universal) properties with $T$. The challenge is then to show that the "existentially closed" $H$-fields are exactly the $H$-fields that share certain deeper first-order properties with $T$. If we can achieve this, then $T$ will be model complete.

An $H$-field $K$ is existentially closed if every differential polynomial over $K$ with a zero in an $H$-field extension of $K$ has a zero in $K$. 
$H$-fields

Let $K$ be an ordered differential field, and put

\[ C = \{ a \in K : a' = 0 \} \] (constant field of $K$)

\[ \mathcal{O} = \{ a \in K : |a| \leq c \text{ for some } c \in C^>^0 \} \] (convex hull of $C$ in $K$)

\[ m(\mathcal{O}) = \{ a \in K : |a| < c \text{ for all } c \in C^>^0 \} \] (maximal ideal of $\mathcal{O}$)

We call $K$ an $H$-field if the following conditions are satisfied:

(H1) \[ \mathcal{O} = C + m(\mathcal{O}), \]

(H2) \[ a > C \implies a' > 0, \]

(H3) \[ a \in m(\mathcal{O}) \implies a' \in m(\mathcal{O}). \]

Examples of $H$-fields: Hardy fields containing $\mathbb{R}$ such as $\mathbb{R}(x, e^x)$, the ordered differential field $\mathbb{R}((x^{-1}))$ of Laurent series, $\mathbb{T}$. 
Liouville closed $H$-fields

The real closure of an $H$-field is again an $H$-field. Call an $H$-field $K$ *Liouville closed* if it is real closed and each differential equation $y' = ay + b$ with $a, b \in K$ has a solution in $K$. For example, $\mathbb{T}$ is Liouville closed. A *Liouville closure* of an $H$-field $K$ is a minimal Liouville closed $H$-field extension of $K$.

**Theorem**

*Each $H$-field has exactly one or exactly two Liouville closures.*

Whether we have one or two Liouville closures is controlled by a key trichotomy in the class of $H$-fields. We discuss this in the next slide.
Trichotomy for $H$-fields

A key feature of any $H$-field $K$ is its valuation $v$ whose valuation ring is the convex hull $\mathcal{O}$ of $C$. Let $\Gamma$ be the value group of $v$ and $\Gamma^* := \Gamma \setminus \{0\}$. The derivation of $K$ induces a function

$$\gamma = v(a) \mapsto \gamma' = v(a') : \Gamma^* \to \Gamma$$

and we put $\Gamma^\dagger := \{\gamma' - \gamma : \gamma \in \Gamma^*\}$. Then $\Gamma^\dagger < (\Gamma^{>0})'$, and exactly one of the following holds:

1. $\Gamma^\dagger < \gamma < (\Gamma^{>0})'$ for some (necessarily unique) $\gamma$;
2. $\Gamma^\dagger$ has a largest element;
3. $\sup \Gamma^\dagger$ does not exist.

If $K = C$ we are in case 1, $\mathbb{R}((x^{-1}))$ falls under case 2, and Liouville closed $H$-fields under case 3. In case 1 there are two Liouville closures of $K$, and in case 2 there is only one.
Immediate Extensions of $H$-fields

For a long time we couldn’t prove that every $H$-field has a case 1 extension. We only knew it for *maximally valued* $H$-fields in case 3. But two years ago we showed:

**Theorem**
Every real closed $H$-field falling under case 3 has an immediate $H$-field extension that is maximally valued.

Complication: such an extension is not in general unique.

**Corollary**
Each $H$-field has a case 1 extension (and thus a case 2 extension).
Consequences for existentially closed $H$-fields

Using the theorem on the previous slides, many known results about $\mathbb{T}$ can now be shown to go through for existentially closed $H$-fields. For example, maximally valued $H$-fields are differentially henselian, and it follows that existentially closed $H$-fields are also differentially henselian. The definition of “differentially henselian” is not so obvious, and involves linear differential operators.

A linear differential operator over a differential field $K$ is an operator $a_0 + a_1 D + \cdots + a_n D^n$ on $K$, where all $a_i \in K$, and $D$ stands for the derivation operator. They form a ring under composition, with $Da = aD + a'$ for $a \in K$. 
Linear differential operators

Let $K$ be an $H$-field and $A = a_0 + a_1 D + \cdots + a_n D^n$ a linear differential operator over $K$, $n \geq 1$, $a_n \neq 0$.

$$v(A) := \min_i va_i.$$ 

**Theorem**
The operator $A$ induces an increasing bijection $A_v : \Gamma \rightarrow \Gamma$ given by $A_v(va) = v(Aa)$, $a \in K^\times$.

**Theorem**
If $K$ is existentially closed, then $A : K \rightarrow K$ is surjective, and $A$ is a product (composition) of operators $a + bD$ of order 1 and operators $a + bD + cD^2$ of order 2.

Both theorems were previously known for $K = \mathbb{T}$. 
Definition of differentially henselian

An H-field $K$ is **differentially henselian** if it has the following property: Let $P(Y) \in \mathcal{O}\{Y\}$ and $a \in \mathcal{O}$, so

$$P(a+Y) = P(a)+a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)} + \text{terms of degree } \geq 2,$$

and suppose that $P(a) \neq 0$, $P(a) \in m(\mathcal{O})$, and $\min \nu a_i = 0$. Let $A := a_0 + a_1 D + \cdots + a_n D^n$, and take the unique $\gamma$ such that $A \nu(\gamma) = \nu(P(a))$. Then there is $b \in \mathcal{O}$ such that $P(b) = 0$ and $\nu(a - b) = \gamma + \delta$, with $m\delta < \nu(P(a))$ for all $m$. 
Set \( a^\dagger := \frac{a'}{a} \), the logarithmic derivative of \( a \).

In \( T \) we consider the sequence \( (\ell_n) \) with

\[
\ell_0 = x, \quad \ell_{n+1} = \log \ell_n.
\]

This sequence is coinitial in \( T^{>\mathbb{R}} \), and

\[
-\ell_n^{\dagger\dagger} = \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \cdots + \frac{1}{\ell_0 \ell_1 \cdots \ell_n}.
\]

Then \( (-\ell_n^{\dagger\dagger}) \) is a pc-sequence without a pseudolimit in \( T \). (It does have a pseudolimit \( \sum_{n=0}^{\infty} \frac{1}{\ell_0 \ell_1 \cdots \ell_n} \) in an \( H \)-field extension of \( T \).)
Another important pseudocauchy sequence

Set $\varrho(b) := (b^\dagger)^2 - 2(b^\dagger)'$. Then

$$\varrho(\ell_n^\dagger) = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2 \ell_1^2} + \cdots + \frac{1}{\ell_0^2 \ell_1^2 \cdots \ell_n^2}$$

also gives a pc-sequence without pseudolimit in $\mathbb{T}$. These facts can be converted into elementary properties of $\mathbb{T}$ that seem to be key to further model-theoretic analysis:

(A1) $\forall a \exists b \left[ v(a - b^\dagger) \leq vb < (\Gamma^{>0})' \right]$;

(A2) $\forall a \exists b \left[ v(a - \varrho(b)) \leq 2vb, \quad vb < (\Gamma^{>0})' \right]$.

A trouble-free $H$-field is one that is real closed, in case 3, and satisfies (A1) and (A2).
Every existentially closed $H$-field is trouble-free.

**Theorem**

Let $K$ be a trouble-free $H$-field and $P \in K\{Y\}$, $P \neq 0$. Then there are $\alpha \in \Gamma$, $a \in K^>\mathbb{C}$ and $m, n \in \mathbb{N}$ such that

$$C < y < a \iff v(P(y)) = \alpha + mvy + nvy'$$

for all $y$ in all $H$-field extensions of $K$.

**Conjecture**: if $K$ is a trouble-free $H$-field, then it has a unique maximal immediate trouble-free $H$-field extension.