Functions that Underlie First-Order Logic

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Four kinds of functions on sequences of finite length

Let U be any nonempty set. It shall serve as base set or also as alphabet. For any n, $0 \leq n < \omega$, ${}^nU$ shall be the set of sequences (words) $x = \langle x_0, \ldots, x_{n-1} \rangle$ of length n such that every x_m is an element (letter) of U. Thus, in particular, the null sequence (null word) $\emptyset$ is the only element of ${}^0U$. For any m, $0 \leq m < \omega$, ${}^{[m,\omega)}U$ will be the set $\bigcup\{{}^nU : m \leq n < \omega\}$. Thus, in particular, ${}^{[0,\omega)}U$ will be the set of all sequences (words) of finite length whose elements (letters) are in U. Concatenation of sequences (words) x and x' in ${}^{[0,\omega)}U$ will be denoted by $x ^\frown x'$.

Consider any i, $0 \leq i < \omega$. ***Excision at place*** i, or i***-excision***, shall be the (unary) function f_i such that $Do\, f_i = {}^{[i+1,\omega)}U$ and, for any x in ${}^nU \subseteq Do\, f_i$,

$$f_i(\langle x_0, \ldots, x_{i-1} \rangle ^\frown \langle x_i \rangle ^\frown \langle x_{i+1}, \ldots, x_{n-1} \rangle) = \langle x_0, \ldots, x_{i-1} \rangle ^\frown \langle x_{i+1}, \ldots, x_{n-1} \rangle \ .$$

For example, if $x = \langle b, e, a, r \rangle = bear$, then $f_0(x) = ear$, $f_1(x) = bar$, $f_2(f_3(x)) = f_2(f_2(x)) = be$, and x is not in $Do\, f_4$. Let $(i, i+1)$-***interchange*** be the function g_i such that $Do\, g_i = {}^{[i+2,\omega)}U$ and, for any x in ${}^nU \subseteq Do\, g_i$,

$$g_i(\langle x_0, \ldots, x_{i-1} \rangle ^\frown \langle x_i, x_{i+1} \rangle ^\frown \langle x_{i+2}, \ldots, x_{n-1} \rangle) = \langle x_0, \ldots, x_{i-1} \rangle ^\frown \langle x_{i+1}, x_i \rangle ^\frown \langle x_{i+2}, \ldots, x_{n-1} \rangle \ .$$

For example, if $x = bear$, then $g_2(g_1(bear)) = bare$. The function $g_i^{=} = \{\langle x, g_i(x) \rangle : g_i(x) = x\}$ shall be ***the restriction of*** g_i ***to its set of fixed points.*** For example, if $x = beer$, then $g_1^{=}(x) = x = beer$, whereas neither $g_0^{=}$ nor $g_2^{=}$ is defined for x. Note that $\{g_i^{=}(x) : x \in {}^{[0,\omega)}U\} = \{\langle x_0, \ldots, x_{n-1} \rangle \in {}^{[i+2,\omega)}U : x_i = x_{i+1}\}$. As usual, for binary relations R and S, let $R^{\smile} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ and $R \circ S = \{\langle x, z \rangle : \langle x, y \rangle \in R \text{ and } \langle y, z \rangle \in S \text{ for some } y\}$. Thus $\langle x, x' \rangle$ is in $f_i \circ f_i^{\smile}$ if and only if for some n, $i < n < \omega$, x and x' both are in ${}^nU$, and $x_j = x'_j$ if $j \neq i$. I let i-***fusion*** be the binary function h_i such that $Do\, h_i = f_i \circ f_i^{\smile}$ and, for any $\langle x, x' \rangle$ in $Do\, h_i$,

$$h_i(x, x') = \langle x_0, \ldots, x_{i-1} \rangle ^\frown \langle x_i, x'_i \rangle ^\frown \langle x_{i+1}, \ldots, x_{n-1} \rangle \ .$$

For example, if $x = bet$ and $x' = bat$, then $h_1(x, x') = beat$, $h_1(x', x) = baet$, and $h(x, x) = beet$, while h_0 and h_2 are not defined for $\langle x, x' \rangle$.

Among these functions, certain ones can be defined in terms of certain others. Among others there hold, as one can verify, the following definabilities.

Theorem 1 *For any set $U \neq \emptyset$ and any i, $0 \leq i < \omega$, let $f = f_i$, $f' = f_{i+1}$, $f'' = f_{i+2}$, $g = g_i$, $g^{=} = g_i^{=}$, and $h = h_i$. Then there hold the following equalities.*

D1. $g = (f \circ f'^{\smile}) \cap (f' \circ f^{\smile})$.

D2. $f' = g \circ f$.

D3. $g^{=} = g \cap (g \circ g)$.

D4. $h = \{\langle x, x', w\rangle : f(x) = f(x'),\ f(w) = x,\ f'(w) = x'\}$.

D5. $f'' = (f \circ f' \circ f^{\smile}) \cap (f' \circ f' \circ f'^{\smile})$. □

From Theorem 1 there follows that, for any $U \neq \emptyset$, each of the functions $f_{i+2},\ g_i,\ g_i^{=},\ h_i,\ 0 \leq i < \omega$, on ${}^{[0,\omega)}U$, is definable from $\{f_0, f_1\}$. Also, for example, each of the functions $f_{i+1},\ g_i^{=},\ h_i,\ 0 \leq i < \omega$, is definable from f_0 and $\{g_i : 0 \leq i < \omega\}$.

Some properties of excision

Let f and f' be any unary functions, i.e., binary relations that are single-valued. Then f' shall be an ***affiliate of*** f, and also $\langle f, f'\rangle$ shall satisfy condition **A**, if and only if

(i) $Do\, f' = Do(f \circ f)$ and

(ii) $f(f'(x)) = f(f(x))$ for any x in $Do\, f'$.

Condition (ii) is equivalent to the condition that, for any x in $Do\, f'$, $f'(x)$ is in $\{y : f(y) = f(f(x))\}$. Thus, roughly speaking, if f' is an affiliate of f, then f' stays close to f.

A pair $\langle f, f'\rangle$ shall satisfy condition **S** if and only if $f \circ f^{\smile} \subseteq f^{\smile} \circ f'$. Note that this condition is equivalent to $f \circ f^{\smile} \subseteq f'^{\smile} \circ f$. Also **S** is equivalent to the condition that f' is in the following sense, ***locally surjective with respect to*** f: For any x in $Do\, f$, if f'_x is the restriction of f' to $\{w : f(w) = x\}$, then f'_x is a surjection from $\{w : f(w) = x\}$ to $\{x' : f(x') = f(x)\}$. This has the following consequence: For any x in $Do\, f$, $\|\{w : f(w) = x\}\| \geq \|\{x' : f(x') = f(x)\}\|$.

Any pair $\{f, f'\}$ of functions, and also any ordered pair $\langle f, f'\rangle$, shall be ***injective*** if and only if the following condition, stated in two ways, is satisfied.

I. If $f(w) = f(x)$ and $f'(w) = f'(x)$, then $w = x$.
I. $(f \circ f^{\smile}) \cap (f' \circ f'^{\smile}) \subseteq \{\langle x, x\rangle : x \in Do\, f \cap Do\, f'\}$.

(Evidently, the condition that results when $\subseteq$ is replaced by $=$ is equivalent.) A consequence of **I** is the following: For any x in $Do\, f$, $\|\{w : f(w) = x\}\| \leq \|\{x' : f(x') = f(x)\}\|$.

Instead of saying that f' is an affiliate of f such that $\langle f, f'\rangle$ satisfies **S**, **I**, or **S** and **I**, respectively, I shall also say that f' is an **S-*affiliate***, **I-*affiliate***, or $\{\mathbf{S}, \mathbf{I}\}$***-affiliate*** of f, respectively. The following is worth noting: If f has an $\{\mathbf{S}, \mathbf{I}\}$-affiliate, then for any x in $Do\, f$, $\|\{w : f(w) = x\}\| = \|\{x' : f(x') = f(x)\}\|$. (Thus if f has an $\{\mathbf{S}, \mathbf{I}\}$-affiliate, then the directed graph picturing f has the following property: For any x in $Do\, f$, x and $f(x)$ have the same, in-degree.)

Of the following two conditions on a function f, the second is non-elementary.

R. z is in $Rg\, f \cap -Do\, f$.
T. For every x in $Rg\, f$ there is some n, $0 \leq n < \omega$, necessarily unique, such that $f^n(x) = z$.

Any z satisfying **R** shall be a ***root of*** f. If there is some z such that **R** holds, then f shall be ***rooted***. If both **R** and **T** hold, then the mono-unary partial algebra $\langle Rg\, f, f, z\rangle$ shall be a ***rooted tree***. In that case, z is the only element of $Rg\, f \cap -Do\, f$. Also, if for any n, $0 \leq n < \omega$, one lets $V_n = \{x : f^m(x) = z \text{ for some } m,\ 0 \leq m \leq n\}$. then $Rg\, f = \bigcup\{V_n : 0 \leq n < \omega\}$. Moreover, by condition **R**, $V_0 \neq \emptyset$ and $V_1 \cap -V_0 \neq \emptyset$.

For any cardinal κ, a function f shall be ***κ-regular*** if and only if $Do\, f \subseteq Rg\, f$ and, for any y in $Rg\, f$, $\|\{x : f(x) = y\}\| = \kappa$. If f is κ-regular for some $\kappa \geq 1$, then f shall be ***regular***. There follows that if $\langle Rf\, f, f, z\rangle$ is a rooted tree and f' is an $\{\mathbf{S}, \mathbf{I}\}$ affiliate of f, then f is regular and hence is κ-regular, where $\kappa = \|\{y : f(y) = z\}\|$. Moreover, as can be shown, in this case f' also is κ-regular. In fact, it is a κ-regular "forest", which consists of κ pairwise disjoint κ-regular trees, each of whose roots is in $\{y : f(y) = z\} = V_1 \cap -V_0$.

One can readily verify the following.

Lemma 1 *Let U be any nonempty set and, for any i, $0 \leq i < \omega$, let f_i be i-excision.*

(a) *For any i, $0 \leq i < \omega$, f_{i+1} is an $\{\mathbf{S}, \mathbf{I}\}$ affiliate of f_i.*

(b) *Every f_i is $\|U\|$-regular.*

(c) *$\langle {}^{[0,\omega)}U, f_0, \emptyset\rangle$ is a rooted tree.* □

Axiomatization and representation

Lemma 1(a), for the case $i = 0$, and Lemma 1(c) together yield a condition that is necessary for a bi-unary partial algebra (with a distinguished element) to be isomorphic] to an algebra $\langle {}^{[0,\omega)}U, f_0, f_1, \emptyset\rangle$. According to the following theorem, the condition also is sufficient.

Theorem 2 *Consider any bi-unary partial algebra $\mathbf{V} = \{Rg\, f, f, f', z\rangle$ such that $\langle Rg\, f, f, z\rangle$ is a rooted tree and f' is an $\{\mathbf{S}, \mathbf{I}\}$ affiliate of f. Let $U = \{y : f(y) = z\}$ and let ϕ be the bijection from U to ${}^1U$ such that for any y in U, $\phi(y) = \langle y\rangle$. Then ϕ can be extended (in a unique way) to an isomorphism from $\langle Rg\, f, f, f', z\rangle$ to $\langle {}^{[0,\omega)}U, f_0, f_1, \emptyset\rangle$.*

Proof. For any n, $0 \leq n < \omega$, let $V_n = \{v \in V : f^m(v) = z \text{ for some } m,\ 0 \leq m \leq n\}$ and let $W_n = \bigcup\{{}^mU : 0 \leq m \leq n\}$. Also, for any n, $0 \leq n < \omega$, let $\mathbf{V}_n = \langle V_n, f, f', z\rangle$ and $\mathbf{W}_n = \langle W_n, f_0, f_1, \emptyset\rangle$ be the subalgebra of $\langle V, f, f', z\rangle$ or of $\langle {}^{[0,\omega)}U, f_0, f_1, \emptyset\rangle$, respectively, whose universe is V_n or W_n, respectively. Then $\phi_1 = \phi \cup \{\langle z, \emptyset\rangle\}$ is an isomorphism from $\mathbf{V}_1$ to $\mathbf{W}_1$. For $1 \leq n < \omega$, assume as inductive hypothesis that ϕ_n is an isomorphism from $\mathbf{V}_n$ to $\mathbf{W}_n$ which includes ϕ_1. To extend ϕ_n to an isomorphism from V_{n+1} to W_{n+1}, consider any v in $V_{n+1} \cap -V_n$ and let $x = f(v)$ and $x' = f'(v)$, so that $\phi_n(x)$ and $\phi_n(x')$ are an element $\langle u_0, \ldots, u_{n-1}\rangle$ or $\langle u'_0, \ldots, u'_{n-1}\rangle$, respectively, of ${}^nU$. Since f' is an affiliate of f, therefore $f(x) = f(x')$.

Since ϕ_n is an isomorphism, therefore $f_0(\langle u_0, \dots u_{n-1}\rangle) = f_0(\phi_n(x)) = \phi_n(f(x')) = f_0(\langle u'_0, \dots, u'_{n-1}\rangle)$. Hence $u_m = u'_m$ for any $m \neq 0$. Let

$$\phi_{n+1}(v) = \langle u'_0, u_0\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle = \langle u'_0, u_0\rangle^\frown \langle u'_1, \dots, u'_{n-1}\rangle \ .$$

There follows

$$f_0(\phi_{n+1}(v)) = \langle u_0, u_1, \dots, u_{n-1}\rangle = \phi_n(x) = \phi_n(f(v)) \ .$$
$$f_1(\phi_{n+1}(v)) = \langle u'_0, u'_1, \dots, u'_{n-1}\rangle = \phi_n(x') = \phi_n(f'(v)) \ .$$

Let ϕ_{n+1} be the function whose domain is V_{n+1} such that, if v is in V_n, then $\phi_{n+1}(v) = \phi_n(v)$ and, if v is in $V_{n+1} \cap -V_n$, then $\phi_{n+1}(v)$ is the element of ${}^{n+1}U$ that is shown above. Now consider any v in $V_{n+1} \cap -V_n$. Since $f(v)$ and $f'(v)$ are in V_n, therefore $\phi_n(f(v)) = \phi_{n+1}(f(v))$ and $\phi_n(f'(v)) = \phi_{n+1}(f'(v))$. From these two equalities and the two displayed above, there now follows that ϕ_{n+1} is a homomorphism from $\mathbf{V}_{n+1}$ to $\mathbf{W}_{n+1}$.

To see that ϕ_{n+1} is surjective, consider any sequence $\langle u', u\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle$ in ${}^{n+1}U$. Then for a unique x' and x in V_n, $\langle u'\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle = \phi_n(x')$ and $\langle u\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle = \phi_n(x)$. Since $f_0(\langle u'\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle) = f_0(\langle u\rangle^\frown \langle u_1, \dots, u_{n-1}\rangle)$ and since ϕ is an isomorphism from $\mathbf{V}_n$ to $\mathbf{W}_n$ therefore $f(x')) = f(x)$. Since $\langle f, f'\rangle$ satisfies $\mathbf{S}$, therefore there is some v in V_{n+1} such that $f(v) = x'$ and $f'(v) = x$. Then $\phi_{n+1}(v) = \langle u', u\rangle^\frown \langle u_1, \dots u_{n-1}\rangle$. There follows that ϕ_{n+1} is a surjection from V_{n+1} to W_{n+1}. Finally, since $\{f, f'\}$ is an injective pair, therefore ϕ_{n+1} is injective. There now follows that $\bigcup\{\phi_n : 0 \leq n < \omega\}$ is an isomorphism from $\mathbf{V}$ to $\mathbf{W}$. □

A consequence of Theorem 2 is the following. Algebras $\mathbf{V}$ and $\mathbf{V}'$ satisfying the conditions of the theorem are isomorphic if and only if $\|V \cap -\{z\}\| = \|V' \cap -\{z'\}\|$. From the proof of the theorem one can also see that any bijection from $V_1 \cap -\{z\}$ to $V_1 \cap -\{z\}$ can be extended to an automorphism of $\mathbf{V}$ and that these are the only automorphisms of $\mathbf{V}$.

Definitional expansion. Excision algebras

Consider any algebra $\mathbf{V} = \langle {}^{[0,\omega)}U, f_0, f_1, \emptyset\rangle$ where U is any nonempty set. Its expansion $\mathbf{V}^1 = \langle {}^{[0,\omega)}U, f_0, f_1, g_0, g_0^=, h_0, \emptyset\rangle$ shall be a ***definitional expansion of*** $\mathbf{V}$, since by Theorem 1 there holds for it $\mathbf{D1}, \mathbf{D3}$, and $\mathbf{D4}$. The expansion $\mathbf{V}^2 = \langle {}^{[0,\omega)}U, f_0, f_1, f_2, g_0, g_1, g_0^=, g_1^=, h_0, h_1, \emptyset\rangle$ of $\mathbf{V}^1$, in turn can be obtained from $\mathbf{V}^2$ by first using $\mathbf{D5}$ and then $\mathbf{D1}, \mathbf{D3}$ and $\mathbf{D4}$. From V^2 in turn by thus using $\mathbf{D5}$, $\mathbf{D1}$, $\mathbf{D3}$, $\mathbf{D4}$ altogether ω times one obtains the algebra $\mathbf{V}^3 = \langle {}^{[0,\omega)}U, f_i, g_i, g_i^=, h_i, \emptyset\rangle_{i<\omega}$. It shall be a $\{\mathbf{D1}, \mathbf{D3}, \mathbf{D4}, \mathbf{D5}\}$ ***expansion of*** $\mathbf{V}$, and also of $\mathbf{V}^1$ and of $\mathbf{V}^2$. Now consider any algebra $\mathbf{V}' = \langle Rg\, f'_0, f'_0, f'_1, z\rangle$ which is isomorphic to $\mathbf{V} = \langle {}^{[0,\omega)}U, f_0, f_1, \emptyset\rangle$. From Theorem 1 there follows that it has a $\{\mathbf{D1}, \mathbf{D3}, \mathbf{D4}, \mathbf{D5}\}$ expansion $\mathbf{V}'' = \langle Rg\, f'_0, f'_i, g'_i, g_i'^=, h'_i, z\rangle_{i<\omega}$ which is isomorphic to $\mathbf{V}^3 = \langle {}^{[0,\omega)}U, f_i, g_i, g_i^=, h_i, \emptyset\rangle_{i<\omega}$.

An ***excision algebra*** shall be any algebra $\mathbf{V}$ such that some definitional expansion of $\mathbf{V}$ is, for some $U \neq \emptyset$, isomorphic to $\langle {}^{[0,\omega)}U, f_i, g_i, g_i^=, h_i, \emptyset\rangle_{i<\omega}$. If, for some $U \neq \emptyset$, the universe of $\mathbf{V}$ is the set ${}^{[0,\omega)}U$, then $\mathbf{V}$ shall be ***based on*** U. Thus, among others, each of the above algebras $\mathbf{V}$, $\mathbf{V}^1$, $\mathbf{V}^2$, $\mathbf{V}^3$ is an excision algebra based on the same set U. Also, for example, by Theorems 1 and 2, the above algebras $\mathbf{V}'$ and $\mathbf{V}''$

are an excision algebra. The theories of each of these six algebras are ***definitionally equivalent*** in the sense that by suitable uses of $\mathbf{D1}, \mathbf{D3}, \mathbf{D4}, \mathbf{D5}$ each can be extended to a theory whose set of theorems is the same as the set of theorems of the theory of $\mathbf{V}^3$, which is also that of the theory of $\mathbf{V}^2$.

Different ones among these theories have different advantages and disadvantages. An obvious advantage of the theory of an algebra such as $\mathbf{V}' = \langle Rg\, f, f, f', z\rangle$ is that it involves only two functions, each of which, moreover, is unary, and only five axioms which somehow seem quite natural by themselves as well as in conjunction with one or more of the others. These five axioms also are fairly easy to visualize, by using, along with dots, arrows of two kinds or colors. The **D5** expansion $\langle Rg\, f'_0, f'_i, z\rangle_{i<\omega}$ of $\langle Rg\, f'_0, f'_1, z\rangle$ brings out the similarity of the algebras $\langle Rg\, f'_i, f'_i, f'_{i+1}\rangle$ and $\langle Rg\, f'_j, f'_j, f'_{j+1}\rangle$, $i \neq j$, and also how they are related. However **D5** may turn out to be less tractable than $\mathbf{D1}, \mathbf{D2}, \mathbf{D3}$, or **D4**.

Use of **D5** can be avoided by use of $\{f_0\} \cup \{g_i : i < \omega\}$ as a set of primitive functions and repeated use of **D2**. As one can see, the closure S_p of $\{g_i : i < \omega\}$ under $\circ$ and $\smile$ forms an inverse semigroup $\mathbf{S}_p = \langle S_p, \circ, \smile\rangle$ which is closely related to the group $\langle G, \circ, \smile\rangle$, where G is the closure under $\circ$ of the set $\{(i, i{+}1) : 0 < i < \omega\}$ of transpositions on $\omega = \{n : 0 \leq n < \omega\}$. A presentation of $\mathbf{S}_p$ is given on pp.157–163 of [C 06]. It could be used as part of a theory of the algebras $\langle {}^{[0,\omega)}U, f_0, g_i, g_0^{=}, h_0, z\rangle_{i<\omega}$.

According to the following lemma the need for using as axiom the condition that $\{f, f'\}$ or $\{f_0, g_0 \circ f_0\}$ is an injective pair (i.e., satisfies condition **I**) can be avoided when dealing, for example, with excision algebras **V** of the kind that is shown there.

Lemma 2 *Consider any algebra* $\mathbf{V} = \langle Rg\, f, f, f', h, z\rangle$ *such that* f' *is an* **S**-*affiliate of* f, $\langle Rg\, f, f, z\rangle$ *is a rooted tree, and* h *is the function defined from* f *and* f' *by* **D4**. *For any* n, $0 \leq n < \omega$, *let* $V_n = \{v \in V : f^m(v) = z$ *for some* $m, 0 \leq m \leq n\}$. *Then the following set of conditions implies that* $\{f, f'\}$ *is an injective pair:* $\{V_{n+1} \cap -V_n\} \subseteq \{h_n(x, x') : \{x, x'\} \subseteq V_n \cap -V_{n-1}\}$ *if* $1 \leq n < \omega$. □

(Note that, since f' is an **S**-affiliate of f, $\subseteq$ may be replaced by $=$.)

Another important way in which the function h_0 complements the functions f_0 and f_1 is brought out by the following lemma.

Lemma 3 *For any* $U \neq \emptyset$, *let* **V** *be the excision algebra* $\langle {}^{[0,\omega)}U, f_0, f_1, h_0, \emptyset\rangle$ *and let* Y *be any nonempty subset of* ${}^{[0,\omega)}U \cap -{}^0U$. *Let* **X** *be the subalgebra of* $\langle {}^{[0,\omega)}U, f, g, \emptyset\rangle$ *that is generated by* Y, *let* $W_1 = X \cap ({}^0U \cup {}^1U)$, *and let* W *be the closure of* W_1 *under* h_0. *Then* W *is the universe of the subalgebra of* **V** *that is generated by* Y. □

A consequence of Lemma 3 is the following: There is a one-one correspondence between the subalgebras of **V** and the nonempty subsets of ${}^1U$.

Operations on sets of sequences of finite length

For any set $V \neq \emptyset$ and any binary relation f on V, f^* shall be the (unary) operation (i.e., total function) on the power set $\{W : W \subseteq V\}$ of V such that, for any $W \subseteq V$, $f^*(W) = \{y : \langle x, y\rangle \in f, \text{ for some } x \in W\}$.

Thus, $f^{\smile *} = (f^{\smile})^*$ is the operation on $\{W : W \subseteq V\}$ such that $f^{\smile *}(W) = \{x : \langle x, y\rangle \in f,\ \text{for some } y \in W\}$. Thus, $f^*(W)$ is the direct image of W under f and $f^{\smile *}(W)$ is the inverse image of W under f. In [JT], the operations f^* and $f^{\smile *}$ are called ***conjugates*** of each other. For any v in V, v^* shall be $\{v\}$.

Thus, if $V = {}^{[0,\omega)}U$ for some $U \neq \emptyset$ then, for any $W \subseteq {}^{[0,\omega)}U$, $W' \subseteq {}^{[0,\omega)}U$, and i, $0 \leq i < \omega$, there hold:

$$
\begin{aligned}
{f_i}^{\smile *}(W) &= \{x^\frown\langle u\rangle^\frown y : x \in {}^iU,\ u \in U,\ x^\frown y \in W\}\ . \\
f_i^*(W) &= \{x^\frown y : x \in {}^iU,\ x^\frown\langle u\rangle^\frown y \in W \text{ for some } u \in U\}\ . \\
g_i^*(W) &= \{x^\frown\langle u, u'\rangle^\frown y : x \in {}^{i-1}U,\ \{u, u'\} \subseteq U,\ x^\frown\langle u', u\rangle^\frown y \in W\rangle\ . \\
(g_i^=)^*(W) &= W \cap \{x \in {}^{[i+2,\omega)}U : x_i = x_{i+1}\}\ . \\
h_i^*(W, W') &= {f_i}^{\smile *}(W) \cap {f_{i+1}}^{\smile *}(W') \\
&= \{x^\frown\langle u, u'\rangle^\frown y : x \in {}^{i-1}U,\ \{u, u'\} \subseteq U,\ x^\frown\langle u\rangle^\frown y \in W,\ x^\frown\langle u'\rangle^\frown y \in W'\}\ . \\
\emptyset^* &= \{\emptyset\} = {}^0U\ .
\end{aligned}
$$

For example, for any $u \neq \emptyset$ and any $W \subseteq {}^2U$,

$$
\begin{aligned}
f_0^*(W) &= \{\langle u'\rangle : \langle u, u'\rangle \in W,\ \text{for some } u \in U\}\ , \\
f_1^*(W) &= \{\langle u'\rangle : \langle u', u\rangle \in W,\ \text{for some } u \in U\}\ .
\end{aligned}
$$

Thus, if one "identifies" $\langle u'\rangle$ and u', then for any $W \subseteq {}^2U$, $f_0^*(W)$ and $f_1^*(W)$ are the range or domain, respectively, of W.

Now assume for example, that U is the set $\mathbb{R}$ of real numbers and that W is a circle in the plane $\mathbb{R}^2$. Then ${f_2}^{\smile *}(W)$ is the cylinder $W \times U = \{\langle u_0, u_1, u\rangle : u \in \mathbb{R},\ \langle u_0, u_1\rangle \in W\}$ that is obtained by drawing an infinite vertical line through every $\langle u_0, u_1\rangle$ in W, while ${f_1}^{\smile *}(W) = \{\langle u_0, u, u_1\rangle : u \in \mathbb{R},\ \langle u_0, u_1\rangle \in W\}$ and ${f_0}^{\smile *}(W) = \{\langle u, u_0, u_1\rangle : u \in \mathbb{R},\ \langle u_0, u_1\rangle \in W\}$. (For similarities and dissimilarities see Figure 1.1.7 in [HMT].)

Again letting $U = \mathbb{R}$, consider any subset W of ${}^2U = \mathbb{R}^2$. Then $g_0^*(W) = \{\langle u, u'\rangle : \langle u', u\rangle \in W\}$. Thus $g_0^*(W)$ results from W by reflection with respect to the line $g_0^{=*}({}^2\mathbb{R}) = \{\langle u, u\rangle : u \in \mathbb{R}\}$.

For the binary operations h_0^*, h_0^*, h_1^*, respectively, there hold, for example:

$$
\begin{aligned}
h_0^*(W, W') &= \{\langle u, u'\rangle : u \in W,\ u' \in W\},\ \text{if } \{W, W\} \subseteq {}^1U\ , \\
h_0^*(W, W') &= \{\langle u_0, u_0', u_1\rangle : \langle u_0, u_1\rangle \in W,\ \langle u_0', u_1\rangle \in W'\},\ \text{if } \{W, W'\} \subseteq {}^2U\ , \\
h_1^*(W, W') &= \{\langle u_0, u_1, u_1', u_2\rangle : \langle u_0, u_1, u_2\rangle \in W,\ \langle u_0, u_1', u_2\rangle \in W'\},\ \text{if } \{W, W'\} \subseteq {}^3U\ .
\end{aligned}
$$

For the element $\emptyset^* = \{\emptyset\} = {}^0U$ of $\{W : W \subseteq {}^{[0,\omega}U\}$ there holds, for any n, $0 \leq n < \omega$,

$$|(f_0^n)^{\smile *}(\emptyset^*) = (f_0^n)^{\smile *}({}^0U)| = {}^nU\ .$$

These examples illustrate that the operations f_i^*, ${f_i}^{\smile *}$, g_i^*, $g_i^{=*}$, h_i^*, $\emptyset^*$ are widely used in mathematical practice and also often in common reasoning.

The following theorem shows that the above set of operations, or any subset of them from which the rest of them is definable, together with the Boolean operations $\cap$, $\cup$, $\bar{\ }_2$, where $\bar{\ }_2$ is relative complementation, has ***adequate expressive power*** with respect to first-order logic (with equality). It is close to being folklore. There is some discussion of it on pp.9–17 of [C 06].

Theorem 3 *Let* $\langle U, W_{k'}\rangle_{k'<k}$ *be any structure such that* $U \neq \emptyset$ *and for every* k', $0 < k' < k$, *there is some* $r_{k'}$, $1 \leq r_{k'} < \omega$, *such that* $W_{k'} \subseteq {}^{r_{k'}}U$. *Consider any subset* W' *of* ${}^{m}U$, *where* $1 \leq m < \omega$. *Then* W' *can be defined in* $\langle U, W_{k'}\rangle_{k'<k}$ *by a formula of first-order logic with equality (but without function symbols or individual constants) if and only if* W' *is in the closure of* $\{W_{k'} : k' < k\}$ *under* $\{\cup, \cap, \bar{\ }_2\}$ *and* $\{f_0^*, f_0^{\smile *}, g_0^{=*}, \{\emptyset\}, g_i^*\}_{i<\omega}$. □

A central role in the theory of cylindric set algebras is played by the operations of i-cylindrification, $0 \leq i < \omega$, which operate on the class $\{W : W \subseteq {}^{\omega}U\}$, where ${}^{\omega}U$ is the set of sequences $x = \langle x_i : 0 \leq i < \omega\rangle$ of length ω such that every x_i is in U. (Cf.[HMT].) For any i, $0 \leq i < \omega$, let b_i be the following equivalence relation on ${}^{[0,\omega)}U$:

$$b_i = \bigcup_{n \leq i} \{\langle x, x\rangle : x \in {}^{n}U\} \cup (f_i \circ f_i^{\smile}) .$$

Then for any i, $0 \leq i < \omega$, an analogue of i-cylindrification on $\{W : W \subseteq {}^{\omega}U\}$ is the operation b_i^* on $\{W : W \subseteq {}^{[0,\omega)}U\}$. For any $W \subseteq {}^{[0,\omega)}U$, it satisfies the equality **D7** in Theorem 4 below.

One can verify that for any $U \neq \emptyset$ there hold among the operations on $\{W : W \subseteq {}^{\omega}U\}$ the following definabilities.

Theorem 4 *For any* $U \neq \emptyset$, $\{W, W'\} \subseteq U$ *and* i, $0 \leq i < \omega$, *there hold the following equalities.*

D2*. $f_{i+1}^*(W) = f_i^*(g_i^*(W))$.

D3*. $g_i^{=*}(W) = g_i^*(W) \cap g_i^*(g_i^*(W))$.

D4*. $h_i^*(W, W') = f_i^{\smile *}(W) \cap f_{i+1}^{\smile *}(W')$.

D6*. ${}^{i}U = f_0^{i\smile *}({}^{0}U) = f_0^{i\smile *}(\emptyset^*)$.

D7*. $b_i^*(W) = (W \cap \bigcup_{j \leq i} {}^{j}U) \cup f_i^{\smile *}(f_i^*(W))$.

For example, to verify **D4***, consider any subsets W and W' of ${}^{[0,\omega)}U$ and any sequence w'' in ${}^{[0,\omega)}U$. Then w'' is in $h_i^*(W, W')$ if and only if for some w in W and some w' in W' there hold the following three equalities: $f_i(w'') = w$, $f_{i+1}(w'') = w'$, and $f_i(w) = f_i(w')$. Since f_{i+1} is an affiliate of f_i, therefore $f_i(w'') = w$ and $f_{i+1}(w'') = w'$ together imply that $f_i(w) = f_i(w')$. Hence the third equality above can be omitted. □

Set algebras. Some steps toward axiomatization.

For any set $U \neq \emptyset$, a ***set algebra based on*** U shall be any algebra $\mathbf{V}$ which is a subalgebra of an algebra $\mathbf{V}'$ whose universe is $\{W : W \subseteq {}^{[0,\omega)}U\}$ such that, if $-$ is unary complementation, then the algebra

$$\mathbf{V}'' = \langle \{W : W \subseteq {}^{[0,\omega)}U\}, \cap, \cup, -, f_i^*, f_i^{\smile *}, g_i^*, g_i^{=*}, h_i^*, {}^iU, b_i^* \rangle_{i<\omega}$$

is a $\{$**D2***,**D3***,**D4***,**D6***,**D7***$\}$ expansion of $\mathbf{V}'$. Thus, for example, for any set $U \neq \emptyset$, any subalgebra $\mathbf{V}$ of the algebra

$$\mathbf{V}''' = \langle \{W : W \subseteq {}^{[0,\omega)}U\}, \cap, \cup, -, f_0^*, f_0^{\smile *}, g_i^*, g_0^{=*}, h_0^*, {}^0U, b_0^* \rangle_{i<\omega}$$

is a set algebra based on U since the above algebra $\mathbf{V}''$ is a $\{$**D2***,**D3***,**D4***,**D6***,**D7***$\}$ expansion of $\mathbf{V}'''$. Also, for example, so is any subalgebra of a **D2*** expansion or of a $\{$**D2***,**D3***,**D6***$\}$ expansion of $\mathbf{V}'''$. A ***set algebra*** shall be any algebra which, for some set $U \neq \emptyset$, is a set algebra based on U. (It shall be a ***full*** set algebra if and only if its universe is the set $\{W : W \subseteq {}^{[0,\omega)}U\}$. Thus the above algebras $\mathbf{V}'$, $\mathbf{V}''$, and $\mathbf{V}'''$ are full set algebras.) Thus, to different choices of operations serving as primitives there correspond different classes of set algebras. Given any class among these there arises the problem of axiomatizing the class of all isomorphic images of members of this class.

(By Theorem 3, unary complementation $-$ is not needed for expressive adequacy; relative complementation $\bar{}_2$ is sufficient. Hence, if one replaces $-$ by $\bar{}_2$ in the above definition of set algebra, one obtains a class of algebras for which questions of axiomatization also are of interest. There is some discussion of this topic in chapter I of [C 06], but it will not be pursued here any further.)

Among aspects relevant to axiomatization, some facts about the operations f_i^* and $f_i^{\smile *}$ will be considered first.

Lemma 4 *Let V be any set, let f and f' be any partial functions on V, and let $f^*, f^{\smile *}, f'^*, f'^{\smile *}$ be the operation on $\{W : W \subseteq V\}$ which, for any $W \subseteq V$ forms the direct image of W under $f, f^{\smile}, f', f'^{\smile}$ respectively.*

(a) *Assume that f' is an affiliate of f. Then:*

A*. $f'^*(V) = (f^2)^*(V)$ *and* $f^*(f'^*(W)) = f^*(f^*(W))$ *for any* $W \subseteq V$.

(b) *Assume that $\langle f, f' \rangle$ satisfies $f \circ f^{\smile} \subseteq f^{\smile} \circ f'$. Then:*

S*. $f^* \circ f^{\smile *} \subseteq f^{\smile *} \circ f'^*$.

(c) *Assume that $Rg\, f \cap -Do\, f = \{z\}$. Then:*

R*. $z^* = \{z\} = f^*(V) \cap -f^{\smile *}(V)$.

(d) *Assume that $\langle V, f, z \rangle$ is a rooted tree. Then:*

T*. $V = \bigcup\{(f^n)^{\smile *}(\{z\}) : 0 \leq n < \omega\}$. □

(For each of the functions $f_i, g_i, g_i^{=}, h_i$ on ${}^{[0,\omega)}U$, there is an analogous function on ${}^{\omega}U$. Likewise, for each of the operations f_i^*, $f_i^{\smile *}$, g_i^*, $g_i^{=*}$, h_i^* on $\{W : W \subseteq {}^{[0,\omega)}U\}$, there is an analogous operation on $\{W : W \subseteq {}^{\omega}U\}$. These functions on ${}^{\omega}U$ and these operations on $\{W : W \subseteq {}^{\omega}U\}$ share many of the formal properties of their analogues. In particular, for them there hold analogues of **D1**,**D2**,**D3**,**D4** or of **D1***,**D3***,**D4***, respectively. Furthermore there hold analogues of **A**, **S**, **I**, or of **A*** and **S*** respectively. Furthermore, instead of considering ${}^{[0,\omega)}U$ and $\{W : W \subseteq {}^{[0,\omega)}U\}$, one may wish to consider for any n, $1 \leq n < \omega$, the sets ${}^0U \cup \cdots \cup {}^nU$ and $\{W : W \subseteq {}^0U \cup \cdots \cup {}^nU\}$ and algebras pertaining to these. For these there also hold many analogues. These two topics also will not be pursued further.)

It is doubtful that for **I** there is a condition that is related to it as **A***,**S***,**R***,**T*** are related to **A**,**S**,**R**,**T** respectively. However, in view of Lemma 2, there is hope that, together with using **T*** and **D6***, one can use **D4*** and the following axiom scheme.

$\mathbf{H}_0^*$ For any i, $1 \leq i < \omega$ and any $W \subseteq {}^{i+1}U$, there are $W' \subseteq {}^iU$ and $W'' \subseteq {}^iU$ such that $W = h_0^*(W', W'')$.

Among the various classes of set algebras that one may wish to axiomatize, I seem to favor, for no very clear reason, the class of algebras whose set of extra-Boolean primitives is the set

$$\{g_i^*\}_{i<\omega} \cup \{f_0^*, \breve{f_0}^*, f_1^*, \breve{f_1}^*, g_0^{=*}, {}^0U\}\ .$$

Then, in axiomatizing the class of isomorphic images of algebras $\langle\{W : W \subseteq {}^{[0,\omega)}U\}, g_i^*\rangle_{i<\omega}$ one would want to bring out the fact that every g_i is a bijection from ${}^{[i+2,\omega)}U$ to ${}^{[i+2,\omega)}U$ such that, moreover, $g_i^*({}^nU) = {}^nU$ for every n, $i+2 \leq n < \omega$. One also would want to make use of some presentation of the inverse semigroup $\mathbf{S}_p$ which was mentioned earlier, since $(g_i \circ g_j)^* = g_i^* \circ g_j^*$, if $0 \leq i \leq j < \omega$. Furthermore, one would try to bring out how $g_0^{=*}$ and g_0^* or,, more generally, $g_i^{=*}$ and g_i^*, are related. The importance of this topic was brought out to me in conversations with Richard Thompson.

There is a sense in which a full set algebra with universe $\{W : W \subseteq {}^{[0,\omega)}U\}$ reflects the structure of the algebra whose universe is ${}^{[0,\omega)}U$ from which it has been obtained. For some choices of primitives this may come out more clearly than for some others. In a change from a full set algebra to its subalgebras some of this information is lost. My hope is that by proceeding in ways illustrated above one will eventually obtain axiomatizations that reflect as closely as possible the structure of the unary algebra of the underlying functions.

Relationship to augmented cylindric set algebras

For any set $U \neq \emptyset$, an ***augmented cylindric set algebra based on*** U shall be any algebra

$$\mathbf{V} = \langle V, \cap, \cup, -, f_0^*, \breve{f_0}^*, g_i^{=*}, {}^iU, b_i^*\rangle_{i<\omega}$$

which is a subalgebra of the algebra $\mathbf{V}'$ whose universe V' is the set $\{W : W \subseteq {}^{[0,\omega)}U\}$. Any algebra $\mathbf{V}$ which, for some $U \neq \emptyset$, is an augmented cylindric set algebra based on U shall be an ***augmented cylindric set algebra***.

Augmented cylindric set algebras are one of the two main topics of chapter 6 of [C 74]. The notion was suggested to me by the use of neat embeddings in the theory of ω-dimensional cylindric algebras (cf [HMT], 400ff). An analogue of the operation $f_0^{\smile *}$ or f_0^*, respectively, on $\{W : W \subseteq {}^{[0,\omega)}U\}$ is the operation Q or P, respectively, on $\{W : W \subseteq {}^{\omega}U\}$ which satisfies, for any $W \subseteq {}^{\omega}U$, the following condition respectively:

$$Q(W) = \{\langle u\rangle^\frown y : u \in U,\, y \in W\}\ .$$
$$P(W) = \{y : \langle u\rangle^\frown y \in W \text{ for some } u \in U\}\ .$$

In [HMT], Q serves as an operation which neatly embeds certain ω-dimensional cylindric algebras in others. In order to utilize certain properties of neat embeddings, it seemed natural, as was done in chapter 6 of [C 74], to make use of the analogues f_0^* and $f_0^{\smile *}$ of Q or P, respectively, in constructing algebras whose universe is a subset of $\{W : W \subseteq {}^{[0,\omega)}U\}$. (Earlier in [B], Bernays used analogues of Q and P in a rather similar way.)

A generalization of the functions $g_i^=$ on a set ${}^{[0,\omega)}U$ of sequences (words) of finite lengths are the following functions $g_{i,j}^=$ where $0 \le i < j < \omega$:

$$g_{i,j}^= = \{\langle x, x\rangle : x \in {}^{[j+1,\omega)}U,\ x_i = x_j\}\ .$$

In Theorem 5(a) below, certain definabilities are given among functions on ${}^{[0,\omega)}U$. The corresponding definabilities among the corresponding operations on $\{W : W \subseteq {}^{[0,\omega)}U\}$, which follow from them, are given in Theorem 5(b). A related theorem is Theorem 6 on pp.22–23 of [C 06]. A proof of **D9** occurs as part of a proof of Theorem 1(a), pp.9–10 of [C 06].

Theorem 5 (a) *For any set $U \neq \emptyset$, any i, $0 \le i < \omega$, and any f_i, f_{i+1}, f_{i+2}, $f_i^{\smile}$, $g_i^=$, $g_{i+2}^=$, b_{i+1} there hold the following definabilities*

D8. $g_{i,i+2}^= = g_i^= \circ f_i^{\smile}$.

D9. $g_i = f_i^{\smile} \circ g_{i,i+2}^= \circ f_{i+2}$.

D10. $f_{i+1} = b_{i+1} \circ g_i^= \circ f_i$.

(b) *For any set $U \neq \emptyset$, any i, $0 \le i < \omega$, and any f_i^*, f_{i+1}^*, f_{i+2}^*, $f_i^{\smile *}$, $g_i^{=*}$, $g_{i,i+2}^{=*}$, b_{i+1}^*, there hold the following definabilities.*

D8*. $g_{i,i+2}^{=*} = g_i^{=*} \circ f_i^{\smile *}$.

D9*. $g_i^* = f_i^{\smile *} \circ g_{i,i+2}^{=*} \circ f_{i+2}^*$.

D10*. $f_{i+1}^* = b_{i+1}^* \circ g_i^{=*} \circ f_i^*$.

To verify **D10** consider any $v = \langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i, x_{i+1}\rangle^\frown y$ in $Do\, b_{i+1} = {}^{[i+2,\omega)}U$. Then $b_{i+1}^*(\{v\}) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i, u\rangle^\frown y : u \in U\}$, hence $g_i^{=*}(b_{i+1}^*(\{v\})) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i, x_i\rangle^\frown y\}$, and hence $f_i^*(g_i^{=*}(b_{i+1}^*(\{v\}))) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i\rangle^\frown y\} = f_{i+1}^*(\{v\})$. □

For any set $U \neq \emptyset$, consider any augmented cylindric set algebra $\mathbf{V}'$ based on U. Since f_0^*, $f_0^{\smile *}$, and every $g_i^{=*}$ and b_i^* are among the primitive functions of $\mathbf{V}$, there follows by induction that every f_i^*, $f_i^{\smile *}$ can be defined using **D10***. Since every $g_i^=$ is among the primitive functions of **V** one can then define every $g_{i,i+2}^{=*}$ using **D8***, and then every g_i^*, using **D9***. Thus, the following set algebra is a {**D8***,**D9***,**D10***} expansion of **V**:

$$\mathbf{V}' = \langle V, \cap, \cup, -, f_i^*, f_i^{\smile\,*}, g_i^*, g_i^{=*}, {}^iU, b_i^* \rangle_{i<\omega} \ .$$

In chapter 6 of [C 74], an axiomatization is given of the class of the algebras $\mathbf{V}''$ which are the isomorphic image of an augmented cylindric set algebra based on some set U. There follows that this axiomatization, when supplemented by **D8***,**D9***,**D10***, is an axiomatization of the class of algebras discussed in the preceding section. For reasons indicated there, other axiomatizations, perhaps with a different set of primitives, corresponding to a different set of underlying functions or relations on ${}^{[0,\omega)}U$, may have advantages.

For any set of extra Boolean primitives, such as the one just described, one can construct in the usual way an algebraic language with symbols for these primitives, symbols for the Boolean operations, and a symbol for equality. One can then define in the usual way a relation $\models$ such that, if $s = t$ is an equality between terms and E is a set of such equalities, then $E \models s = t$ if and only if every model of E, under the interpretation of $\{\cap, \cup, -\}$, $\{g_i^*\}_{i<\omega}$, and $\{f_0^{\smile\,*}, f_0^*, g_0^{=*}, h_0^*, {}^0U\}$ that I have been using, is also a model of $s = t$. Any set E of equalities thus gives rise to a congruence relation on the set of terms and then to an algebra in which to each of the symbols for the above operations there is assigned a function on the resulting congruence classes of terms. (Cf. [HMT], pp.168–170.) The resulting algebra shall be an ***algebra of theories***. For augmented cylindric algebras, their algebras of theories were characterized in the second half of chapter 6 of [C 74], making use of ideas in the unpublished thesis [Ho]. It is likely that methods used there will yield characterizations of theories based on a set of primitives which differs from the one used in [C 74], such as one of the sets of primitives mentioned in the previous section.

The set of equalities that hold in every augmented cylindric set algebra (of operations on sets of sequences of finite length) has been axiomatized in chapter 5 of [C 74]. This allows one to treat certain problems of provability or non-provability in first-order logic with equality as problems concerning an equational theory. There is a fair chance that, by adapting some steps in chapters 4 and 5 of [C 74], one can find for some of the set algebras discussed earlier, an axiomatization of the equalities that hold in these.

Addendum

Theorem 6 *For any set $U \neq \emptyset$ and any i, $0 \leq i < \omega$, there hold:*

D11. $g_{i+1} = (f_i \circ g_i \circ f_i^{\smile}) \cap (f_{i+1}^2 \circ (f_{i+1}^2)^{\smile})$.

D11*. $g_{i+1}^* = (f_i^* \circ g_i^* \circ f_i^{\smile\,*}) \cap (f_{i+1}^2)^* \circ (f_{i+1}^2.)^{\smile\,*}$.

For proof of **D11*** consider any W in $Do\, g_{i+1}^* = \{W : W \subseteq {}^{[i+3,\omega)}U\}$ and any $x = \langle x_0, \ldots, x_{n-1}\rangle$ in ${}^nU$, $i{+}3 \leq n < \omega$. Then:

$$f_i^{\smile\,*}(g_i^*(f_i^*(\{x\}))) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle u\rangle^\frown\langle x_{i+2}x_{i+1}\rangle^\frown\langle x_{i+3}, \ldots, x_{n-1} >: u \in U\}$$

$$(f_{i+1}^2)^{*\smile}((f_{i+1}^2)^*(\{x\})) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i\rangle^\frown\langle u', u''\rangle^\frown\langle x_{i+3}, \ldots, x_{n-1}\rangle : \{u', u''\} \subseteq U\}$$

Hence

$$(f_i^* \circ g_i^* \circ f_i^{\smile\,*})(\{x\}) \cap ((f_{i+1}^2)^* \circ (f_{i+1}^2)^{*\smile}(\{x\}) = \{\langle x_0, \ldots, x_{i-1}\rangle^\frown\langle x_i\rangle^\frown\langle x_{i+2}, x_{i+1}\rangle^\frown\langle x_{i+3}, \ldots, x_{n-1}\rangle\} \ .$$

□

Consider any excision algebra $\mathbf{V} = \langle {}^{[0,\omega)}U, f_0, f_1 \rangle$. The algebra $\mathbf{V}' = \langle {}^{[0,\omega)}U, f_0, f_1, g_0, g_1 \rangle$ is a {**D1,D11**} expansion of $\mathbf{V}$ and the algebra $\mathbf{V}'' = \langle {}^{[0,\omega)}U, f_0, f_1, g_0, g_1, f_2 \rangle$ is a {**D2**} expansion of $\mathbf{V}'$. Then the algebra $\mathbf{V}''' = \langle {}^{[0,\omega)}U, f_0, f_1, f_2, g_0, g_1, g_2 \rangle$ is a **D11** expansion of $\mathbf{V}''$. Thus using **D2** and then **D11** altogether ω times one obtains as a {**D1,D2,D11**} expansion of $\mathbf{V}$ the excision algebra $\mathbf{V}''' = \langle {}^{[0,\omega)}U, f_i, g_i \rangle_{i<\omega}$. Using **D1***, **D2***, and **D11*** in a similar manner one obtains from any set algebra $\langle \{W : W \subseteq {}^{[0,\omega)}U\}, f_0^*, f_0^{\smile *}, f_1^*, f_1^* \rangle$ its {**D1*,D2*,D11***} expansion $\langle \{W : W \subseteq {}^{[0,\omega)}U\}, f_i^*, f_i^{\smile *}, g_i^* \rangle_{i<\omega}$.

References

[B] P.Bernays. Über eine natürliche Erweiterung des Relationskalküls, (in A.Heyting, ed., *Constructivity in Mathematics*). (North-Holland, Amsterdam, 1959), pp.1–14.

[C 74] W.Craig. *Logic in Algebraic Form.* (North-Holland, Amsterdam, 1974), viii+204 pp.

[C 06] W.Craig. *Semigroups Underlying First-Order Logic. Memoirs of the American Math. Society* **806** (2006), xxv + 263 pp.

[Ho] C.M.Howard. "An approach to algebraic logic," Ph.D. Thesis, U. of California, Berkeley (1965).

[HMT] L.Henkin, D.Monk, and A.Tarski. *Cylindric Algebras*, Part I. (North-Holland, Amsterdam, 1974), vi + 508 pp.

[JT] B.Jónsson and A.Tarski. Boolean algebras with operators. Part I. *Am. J. Math.* **73** (1951), 891–939.