Definability in the Computably Enumerable Sets

What I learned from Leo Harrington

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The Computably Enumerable Sets, $\mathcal{E}$

- W_e is the domain of the eth Turing machine.
- $(\{W_e : e \in \omega\}, \subseteq)$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- These sets are the same as the Σ^0_1 sets, $\{x : (\mathbb{N}, +, \times, 0, 1) \models \varphi(x)\}$, where φ is Σ^0_1.
- $W_{e,s}$ is the domain of the eth Turing machine at stage s.
- For safety, all sets are c.e., infinite, and coinfinite, unless otherwise noted.
- $0, 1, \cup, \cap$, and $\sqcup$ (disjoint union) are definable from $\subseteq$ in $\mathcal{E}$.

Computably Isomorphic Sets

Definition

X and Y are *computably isomorphic* iff there is a computable permutation, p, of ω such that $p(X) = Y$.

Lemma

Assume X and Y are computably isomorphic which is witnessed via a computable permutation p. Then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.

Proof.

- If W is c.e. then so is $p(W)$.
- $X \subseteq Y$ iff $p(X) \subseteq p(Y)$.

The first clause depends on the fact that p is computable. The second depends on the fact that p is a permutation. □

1-Complete Sets

Theorem (Myhill)

X is 1-complete iff X and K are computably isomorphic.

Lemma

All 1-complete sets are in the same (effective) orbit.

Question

Do the 1-complete sets form an (effective) orbit?

The Computable Sets

Lemma

The infinite coinfinite computable sets are in the same effective orbit.

Proof.

There is a computable permutation p such that $p(R) = \widehat{R}$ and $p(\overline{R}) = \overline{\widehat{R}}$. □

Lemma

R is computable iff $\exists Y [R \sqcup Y = \omega]$.

Lemma

The infinite coinfinite computable sets form an effective orbit.

1-Complete is Definable

Theorem (Harrington ∼84)

A c.e. set A is 1-complete iff $(\exists C \supset A)(\forall B \subseteq C)(\exists R)[R$ is computable & $R \cap C$ is noncomputable & $R \cap A = R \cap B]$.

Theorem

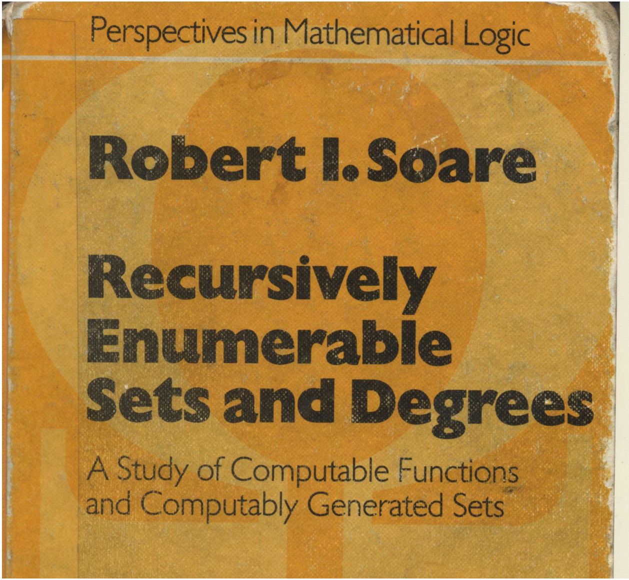
The 1-complete sets form an effective orbit.

Automorphisms vs. Definability

Definition

X is *automorphic* to Y, $X \approx Y$, iff there is an automorphism of $\mathcal{E}$ such that $\Phi(X) = Y$.

If the 1-complete sets had failed to form an orbit then there must be a c.e. set which not 1-complete but is automorphic to a 1-complete set. The failure to find these automorphisms led Leo to the property defining the 1-complete sets. It is this interplay which makes the c.e. sets an interesting place to work.


Perspectives in Mathematical Logic
Robert I. Soare
Recursively Enumerable Sets and Degrees
A Study of Computable Functions and Computably Generated Sets

1.1 Theorem (Harrington). *An r.e. set A is creative iff*

$$(1.1) \qquad (\exists C \supset A)\,(\forall B \subseteq C)\,(\exists R)\,[R \text{ is recursive} \;\&\; R \cap C \text{ is nonrecursive} \;\&\; R \cap A = R \cap B],$$

where all variables range over $\mathcal{E}$.

1.2 Corollary (Harrington). *The property of being creative is elementary lattice theoretic.* ▯

1.3 Definition. (i) Let Aut $\mathcal{E}$ (Aut $\mathcal{E}^*$) denote the group of automorphisms of $\mathcal{E}$ ($\mathcal{E}^*$). The symbols Φ, Ψ will denote automorphisms of $\mathcal{E}$ or $\mathcal{E}^*$.

(ii) For A and $B \in \mathcal{E}$ we say A is *automorphic to* B and write $A \cong_{\mathcal{E}} B$ ($A^* \cong_{\mathcal{E}^*} B^*$) if there exists $\Phi \in$ Aut $\mathcal{E}$ (Aut $\mathcal{E}^*$) such that $\Phi(A) = B$ ($\Phi(A^*) = B^*$).

(iii) The *orbit* of $A \in \mathcal{E}$, written orbit(A), is the class $\{ B : A \cong_{\mathcal{E}} B \}$.

(iv) The *orbit* of $A^* \in \mathcal{E}^*$ is the class $\{ B^* : A^* \cong_{\mathcal{E}^*} B^* \}$.

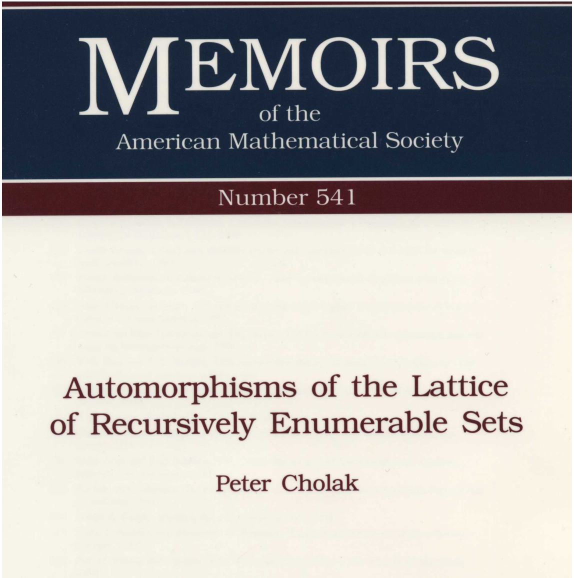
1.4 Corollary (Harrington). *The creative sets constitute an orbit.*

4.6 Theorem (Soare [1974]). *Given any two maximal sets A and B there is an automorphism Φ of $\mathcal{E}$ such that $\Phi(A) = B$.*

Proof. Fix maximal sets A and B. In Theorem 5.1 we shall define skeletons $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ which depend upon A and B, respectively. In Theorem 5.2 we shall then define u.r.e. sequences $\{\hat{U}_n^+\}_{n\in\omega}$ and $\{\hat{V}_n^+\}_{n\in\omega}$ satisfying (4.6) and give a simultaneous enumeration of all the above r.e. sets which satisfies the hypotheses (4.11) and (4.12) of the Extension Theorem. By the conclusion of the Extension Theorem and Corollary 2.9 there exists a 1:1 map p from A to B satisfying (4.7) so $A^* \equiv_{\mathcal{E}^*} B^*$. By Corollary 2.7 $A \equiv_{\mathcal{E}} B$. ▯

1.14 Open Question. For every nonrecursive r.e. set A does there exist $\Phi \in \text{Aut } \mathcal{E}$ such that $\Phi(A) \equiv_T \emptyset'$?

This question appears to be rather difficult. (Some partial results were obtained by Downey and Stob [ta, Theorems 9 and 12] which imply that every low_2 simple set, every simple set A with $\overline{A}$ semi-$\text{low}_{1.5}$, and every d-simple set with a maximal superset is automorphic to a complete set.) The dual question is whether every orbit contains some degree $\mathbf{a} < \mathbf{0}'$. By Theorem XV.1.1 this cannot be exactly true and indeed Harrington claims further that there exists an orbit consisting of only Turing complete but not creative sets. However, Harrington also claims that a revised version of the question has a positive answer. Namely, he claims that if A is r.e., nonrecursive and not Turing complete, then there exists a set B in the orbit of A such that $B \not\leq_T A$. A weaker open question than 1.14 is whether every nonrecursive r.e. set contains *some* high r.e. set in its orbit.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THE Δ_3^0-AUTOMORPHISM METHOD AND NONINVARIANT CLASSES OF DEGREES

LEO HARRINGTON AND ROBERT I. SOARE

1. Introduction

A set A of nonnegative integers is *computably enumerable (c.e.)*, also called *recursively enumerable (r.e.)*, if there is a computable method to list its elements. Let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subseteq)$. Most previously known automorphisms Φ of the structure $\mathcal{E}$ of sets were effective (computable) in the sense that Φ has an effective presentation. We introduce here a new method for generating noneffective automorphisms whose presentation is Δ_3^0, and we apply the method to answer a number of long open questions about the orbits of c.e. sets under automorphisms of $\mathcal{E}$. For example, we show that the orbit of every noncomputable (*i.e.*, nonrecursive) c.e. set contains a set of high degree, and hence that for all $n > 0$ the well-known degree classes $\mathbf{L}_n$ (the low$_n$ c.e. degrees) and $\overline{\mathbf{H}}_n = \mathbf{R} - \mathbf{H}_n$ (the complement of the high$_n$ c.e. degrees) are noninvariant classes.

Chapter XVI

Further Results and Open Questions About R.E. Sets and Degrees

The purpose of this chapter is to give a brief overview without proofs of some further results and current open questions about r.e. sets and r.e. degrees which would have been covered in detail in this book if time and space had permitted. The reader may recognize how the diverse results and methods studied in Chapters VII through XV have been combined and extended in these later theorems. No attempt has been made to be comprehensive, and numerous important and current topics in recursion theory have of necessity been omitted, such as the Turing degrees in general, recursive model theory, effective mathematics, computational complexity, and others.

1. Automorphisms and Isomorphisms of the Lattice of R.E. Sets

Slaman-Woodin Conjecture

Definition

For a c.e. set A, $\mathcal{L}^*(A)$ is $\{W \cup A : W \text{ a c.e. set}\}$ under $\subseteq$ modulo the ideal of finite sets ($\mathcal{F}$) and $\mathcal{E}^*(A)$ is $\{W \cap A : W \text{ a c.e. set}\}$ under $\subseteq$ modulo $\mathcal{F}$.

Theorem (Lachlan (1968))

For each computable Boolean Algebra $\mathcal{B}_i$, there is c.e. set hhsimple H_i such that $\mathcal{L}^*(H_i) \cong \mathcal{B}_i$.

Corollary

The set $\{\langle i, j\rangle : \mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)\}$ is Σ^1_1-complete.

Conjecture (Slaman-Woodin)

The set $\{\langle i, j\rangle : W_i \approx W_j\}$ is Σ^1_1-complete.

Idea: Replace "$\mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)$" with "$W_i \approx W_j$". (Later we will see this fails!)

Invariant Classes

Definition

A class $\mathcal{D}$ of degrees is *invariant* if there is a class $\mathcal{S}$ of (c.e.) sets such that

1. $\mathbf{d} \in \mathcal{D}$ implies there is a W in $\mathcal{S}$ and $\mathbf{d}$.
2. $W \in \mathcal{S}$ implies $\deg(W) \in \mathcal{D}$ and
3. $\mathcal{S}$ is closed under automorphic images (but need not be one orbit).

Conjecture (Martin's Invariance Conjecture)

Among jump classes $\mathbf{H_n}$ *and* $\mathbf{L_n}$, *for* $n > 0$, *and their complements, the invariant classes are exactly* $\mathbf{H_{2n-1}}$ *and* $\overline{\mathbf{L_{2n}}}$.

The Mysteries

Laurie Duggan

The Mysteries

Laurie Duggan

Everything happens at once
We miss most of it.
The kettle boils over
And puts out the fire.

Coding the Double Jump into $\mathcal{E}$

Theorem (Cholak, Harrington)

Let $C = \{\mathbf{a} : \mathbf{a}$ is the Turing degree of a Σ_3 set greater than $\mathbf{0}''\}$. Let $\mathcal{D} \subseteq C$ such that $\mathcal{D}$ is upward closed. Then there is a non-elementary ($\mathcal{L}_{\omega_1,\omega}$) $\mathcal{L}(A)$ property $\varphi_{\mathcal{D}}(A)$ such that $D'' \in \mathcal{D}$ iff there is an A where $A \equiv D$ and $\varphi_{\mathcal{D}}(A)$.

Orbits of Hhsimple Sets

Theorem (Cholak, Harrington)

If A is hhsimple then $A \approx \widehat{A}$ iff $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{L}^*(\widehat{A})$.

Corollary (Cholak, Harrington)

The set $\{\langle i, j\rangle : W_i \approx W_j$ and W_i is hhsimple$\}$ is Σ^0_5.

Hence the Slaman-Woodin plan of attack on their conjecture fails. The proof involves coding (i.e. definability) into $\mathcal{E}$.

Automorphisms to Automorphisms

Theorem (**The Conversion Theorem**, Cholak, Harrington)

If A and $\hat{A}$ are automorphic via Ψ then they are automorphic via Λ where $\Lambda \restriction \mathcal{L}^*(A) = \Psi$ and $\Lambda \restriction \mathcal{E}^*(A)$ is Δ^0_3.

The Scott Rank of $\mathcal{E}$ is $\omega_1^{CK} + 1$

Theorem (Cholak, Harrington)

There is an c.e. set A such that the set

$$\mathcal{I}_A = \{i : A \text{ is automorphic to } W_i\}$$

is Σ_1^1-complete.

Avoiding an Upper Cone

Question (Cone Avoidance)

Given an incomplete c.e. degree $\mathbf{d}$ *and an incomplete c.e. set* A*, is there a* $\hat{A}$ *automorphic to* A *such that* $\mathbf{d} \not\leq_T \hat{A}$*?*

Should we expect an arithmetical answer?

What c.e. sets are automorphic to complete sets?

By Harrington and Soare we know this is related to dynamic properties.

Work with Peter Gerdes and Karen Lange on very tardy sets.

Again should we expect an arithmetical answer?

$\mathcal{D}$-hhsimple Sets

Definition (The sets disjoint from A)

$\mathcal{D}(A) = \{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}$ under inclusion. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$.

Lemma

If A is simple then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{L}^*(A)$.

A is *$\mathcal{D}$-hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Boolean algebra. Except for the creative sets, until recently all known orbits were orbits of $\mathcal{D}$-hhsimple sets.

Question

Are all $\mathcal{D}$-hhsimple sets automorphic to complete sets?

Which sets are automorphic to low sets?

Theorem (Epstein)

There is a properly low$_2$ degree $\mathbf{d}$ *such that if* $A \leq_T \mathbf{d}$ *then* A *is automorphic to a low set.*

Definition (Following Maass)

A has the (Δ^0_3) *low shrinking property* iff for any (Δ^0_3) simultaneous enumeration of the c.e. sets $\{U_e | e \in \omega\}$ we can effectively (Δ^0_3) assign a shrinking U^S_e to each U_e such that $U^S_e \cap \overline{A} =^* U_e \cap \overline{A}$ and for finite F if $\bigcap_{i \in F} U^S_e \cap A$ is infinite then $\bigcap_{i \in F} U_e \cap A$ is infinite (entry states w.r.t. the shrunken sets are the same as the entry w.r.t. given enumeration).

Conjecture (Cholak and Weber)

A is Δ^0_3 automorphic to a low set iff A has the Δ^0_3 low shrinking property.

Thanks, Leo!