Degree of randomness versus Turing degree.

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Motivational Quotes

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- For this talk, P is the paradigm "more random implies computationally weaker"

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- When is P true?
- How is triviality related to the Turing degree?

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- For any given null class (class of measure 0) $\mathcal{C}$, the probability that $A \in \mathcal{C}$ is 0.
- For instance, if we fix a noncomputable set B, the probability that A is Turing incomparable with B is 1.

Randomly Generated Sets are Both Computationally Weak and Complex

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- In fact, a randomly generated set A is almost surely Turing incomparable to every noncomputable arithmetic set (for instance).
- Thus A is computationally weak in the sense that it can't compute any noncomputable arithmetic set.
- But A is computationally complex in the sense that it can't be computed by any arithmetic set.

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- Equivalently, $\forall n K(A \upharpoonright n) \geq^{+} n$.

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- (Kučera) If $\mathbf{a} \geq \mathbf{0}'$ then $\mathbf{a}$ contains an ML-random set. (Contrary to P, there is no limit on the computing power of ML-random sets.)

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- In any Turing degree there are sets that are far from random (in the sense of K-reducibility): Any set can be coded at locations given by the range of a fast-growing order function f. (Contrary to P, there is sequence of sets of increasing randomness with constant computing power.)

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- In any Turing degree there are sets that are far from random (in the sense of K-reducibility): Any set can be coded at locations given by the range of a fast-growing order function f. (Contrary to P, there is sequence of sets of increasing randomness with constant computing power.)
- I don't find it surprising that the Turing degree (a measure of information content) does not determine the K-degree (a measure of data compression).

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- P is true in this context, but A being n-random is not enough to guarantee A is incomparable with all noncomputable arithmetic sets.
- What level of randomness is sufficient? For many purposes ML-randomness (or even pseudo-randomness) is enough.

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- In fact, (Nies) Every K-trivial set is superlow (hence low).
- Contrary to P, ML-random sets can have more computing power than K-trivial sets.

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- Theorem: If $\mathbf{a} \geq \mathbf{0}'$ then $\mathbf{a}$ contains a K_m-trivial set.

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- Easy observation: Every K-trivial is K_m-trivial.
- Theorem: If $\mathbf{a} \geq \mathbf{0}'$ then $\mathbf{a}$ contains a K_m-trivial set.
- However, there are restrictions on the Turing degrees of K_m-trivial sets.

The a-K-trivial and a-K_m-trivial Sets

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- Easy observation: The 1-K-trivial sets are the K-trivial sets (same with K_m).
- Easy observations: The 0-K_m-trivial sets are the computable sets. There are no a-K-trivial sets with $a < 1$.
- Easy observations: Every a-K-trivial set is a-K_m-trivial. If $b > a + 1$, then any a-K_m-trivial set is b-K-trivial.

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- Then for each n, $K(B \upharpoonright n) \leq^+ K(A \upharpoonright f(n)) \leq aK(f(n)) \leq a(K(n) + b)$ for some constant b.

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- Therefore, $K(B \upharpoonright n) \leq^+ aK(n)$.

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- Therefore, $K(B \restriction n) \leq^+ aK(n)$.
- (This proof and the ones below also work for a-K_m-trivials.)

If a computably dominated set A is a-K-trivial, then so is every set Turing reducible to A

- A is said to be computably dominated (or of hyperimmune-free degree) if each function $g \leq_T A$ is dominated by a computable function.

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- (Jockusch, Martin) A is computably dominated iff for all sets B, if $B \leq_T A$ then $B \leq_{tt} A$.
- Thus $A \geq_T B \implies A \geq_{wtt} B \implies B$ is a-K-trivial.

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- There is a computably dominated ML-random set: its degree cannot contain any almost-K-trivial set.
- In particular, it doesn't contain a K_m-trivial set.
- Question: Is there a Δ^0_2 Turing degree that does not contain a K_m-trivial set (or almost-K-trivial set)?

Thanks for listening!

Although he was not involved in this particular project, I would like to thank Leo Harrington on this occasion. Leo was a great Ph.D. advisor and continues to inspire me each time I visit Berkeley.

Thanks also to: Rod Downey and ? for talking to me about hyperimmune-free degrees at the Notre Dame meeting.