# Degrees that are not Degrees of Categoricity

Bernard Anderson and Barbara Csima

University of Waterloo

March 26, 2011

www.math.uwaterloo.ca/$\sim$b7anders

# Computable structures

**Definition**

A structure (coded as a subset of $\omega$) is a computable structure if its domain and atomic diagram are computable.

Without loss of generality, we assume all computable structures have domain $\omega$.

**Notation**

We denote the $n$-th computable structure under some effective listing by $\mathcal{A}_n$.

# Computably categorical structures

**Definition**

Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is computably categorical if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$.

# Computably categorical structures

**Definition**

Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is computably categorical if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$.

**Example**

Given two computable copies of the dense linear orders without endpoints (DLO) we can find a computable isomorphism between them.

Therefore they are computably categorical structures.

# Relatively computably categorical structures

**Definition**

Let $\mathcal{A}$ be a computable structure and $\mathbf{x}$ a Turing degree. We say that $\mathcal{A}$ is **x-computably categorical** if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T \mathbf{x}$.

# Relatively computably categorical structures

## Definition

Let $\mathcal{A}$ be a computable structure and $\mathbf{x}$ a Turing degree. We say that $\mathcal{A}$ is **x-computably categorical** if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T \mathbf{x}$.

## Example

The standard ordering on $\mathbb{N}$ is $\mathbf{0}'$-computably categorical.

To build an isomorphism to a computable copy, we use $\mathbf{0}'$ to determine how many predecessors each element has.

# Degrees of categoricity

**Definition**

$\mathrm{CatSpec}(\mathcal{A}) = \{\mathbf{x} \mid \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}$

# Degrees of categoricity

**Definition**

$\text{CatSpec}(\mathcal{A}) = \{\mathbf{x} \mid \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}$

**Definition (Fokina, Kalimullin, and Miller)**

A Turing degree $\mathbf{x}$ is a degree of categoricity if there is a computable structure $\mathcal{A}$ such that $\mathbf{x} \in \text{CatSpec}(\mathcal{A})$ and for all $\mathbf{y} \in \text{CatSpec}(\mathcal{A})$ we have $\mathbf{x} \leq_T \mathbf{y}$.

Degrees of categoricity are sometimes called categorically definable degrees.

# Degrees of categoricity (continued)

**Summary**

$\mathcal{A}$ witnesses $\mathbf{x}$ is a degree of categoricity if $\mathbf{x}$ is the least degree that can compute isomorphisms between $\mathcal{A}$ and any computable structure isomorphic to it.

**Example**

For example, computable copies of the DLO witness that $\mathbf{0}$ is a degree of categoricity.

# Strong degrees of categoricity

**Definition**

A Turing degree $\mathbf{x}$ is a strong degree of categoricity if there is a computable structure $\mathcal{A}$ with computable copies $\mathcal{B}$ and $\mathcal{M}$ such that $\mathcal{A}$ is $\mathbf{x}$-computably categorical, and for every isomorphism $f : \mathcal{B} \to \mathcal{M}$ we have $\mathbf{x} \leq_T f$.

**Remark**

*Strong degrees of categoricity are degrees of categoricity.*

# Known results (positive)

Fokina, Kalimullin, and Miller developed the basic method for showing degrees are degrees of categoricity.

**Theorem (Fokina, Kalimullin, and Miller)**

*Let* $\mathbf{x}$ *be a d.c.e. degree. Then* $\mathbf{x}$ *is a [strong] degree of categoricity.*

# Known results (positive)

Fokina, Kalimullin, and Miller developed the basic method for showing degrees are degrees of categoricity.

**Theorem (Fokina, Kalimullin, and Miller)**

*Let $\mathbf{x}$ be a d.c.e. degree. Then $\mathbf{x}$ is a [strong] degree of categoricity.*

This result can be relativized to finite and transfinite jumps.

**Theorem (Fokina, Kalimullin, and Miller)**

*Let $n \in \omega$ and let $\mathbf{x}$ be d.c.e.$(\emptyset^{(n)})$ with $\mathbf{x} \geq_T \emptyset^{(n)}$. Then $\mathbf{x}$ is a [strong] degree of categoricity.*

**Theorem (Csima, Franklin, and Shore)**

*Let $\alpha < \omega_1^{CK}$ and let $\mathbf{x}$ be d.c.e.$(\emptyset^{(\alpha)})$ with $\mathbf{x} \geq_T \emptyset^{(\alpha)}$. Then $\mathbf{x}$ is a [strong] degree of categoricity.*

# Known results (negative)

It is easy to see that there are at most countably many degrees of categoricity.

It has been shown that all degrees of categoricity are hyperarithmetical.

**Theorem (Fokina, Kalimullin, and Miller)**

*If $\mathbf{x} \notin HYP$, then $\mathbf{x}$ is not a strong degree of categoricity.*

**Theorem (Csima, Franklin, and Shore)**

*If $\mathbf{x} \notin HYP$, then $\mathbf{x}$ is not a degree of categoricity.*

# Warm up proposition

In this talk we will show several more negative results. We start by considering a straight-forward example.

**Proposition (Anderson and Csima)**

*There is a degree* $\mathbf{x} \leq_T \mathbf{0}''$ *that is not a degree of categoricity.*

# Warm up proposition

In this talk we will show several more negative results. We start by considering a straight-forward example.

**Proposition (Anderson and Csima)**

*There is a degree* $\mathbf{x} \leq_T \mathbf{0}''$ *that is not a degree of categoricity.*

**Ideas for proof**

- We construct a noncomputable $X$ by finite extensions using a $\emptyset''$ oracle.
- We build $X$ so that for any computable structure $\mathcal{A}_m$ we have $\mathrm{Deg}(X) \in \mathrm{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \mathrm{CatSpec}(\mathcal{A}_m)$.

# Warm up proposition (continued)

## Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy:
  Either $\Phi_l^X$ is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.

# Warm up proposition (continued)

## Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy:
  Either $\Phi_l^X$ is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.

- Given a string $\sigma$ we wish to determine if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ cannot be extended to an isomorphism.

# Warm up proposition (continued)

## Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy:
  Either $\Phi_l^X$ is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.
- Given a string $\sigma$ we wish to determine if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ cannot be extended to an isomorphism.
- We ask $\emptyset'$: Is there a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is seen not to be an injective homomorphism?

# Warm up proposition (continued)

## Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy:
  Either $\Phi_l^X$ is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.

- Given a string $\sigma$ we wish to determine if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ cannot be extended to an isomorphism.

- We ask $\emptyset'$: Is there a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is seen not to be an injective homomorphism?

- We ask $\emptyset''$: Is there a $\tau \supseteq \sigma$ and a $d \in \omega$ such that for every $\gamma \supseteq \tau$ we have $d$ is not in the domain or range of $\Phi_l^\gamma$?

# Warm up proposition (continued)

## Ideas for proof (continued)

- For every $(l, m, k)$ we want to satisfy:
  Either $\Phi_l^X$ is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.

- Given a string $\sigma$ we wish to determine if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ cannot be extended to an isomorphism.

- We ask $\emptyset'$: Is there a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is seen not to be an injective homomorphism?

- We ask $\emptyset''$: Is there a $\tau \supseteq \sigma$ and a $d \in \omega$ such that for every $\gamma \supseteq \tau$ we have $d$ is not in the domain or range of $\Phi_l^\gamma$?

- Yes: extend to $\tau$. No: there is a computable isomorphism.

# 2-generic relative to some perfect tree

We wish to generalize this proof to come up with a negative result on a broad class of sets.

**Definition**

A set $G$ is *$n$-generic* if for every $\Sigma_n$ subset $S$ of $2^{<\omega}$ there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ we have $\tau \notin S$.

# 2-generic relative to some perfect tree

We wish to generalize this proof to come up with a negative result on a broad class of sets.

**Definition**

A set $G$ is $n$-generic if for every $\Sigma_n$ subset $S$ of $2^{<\omega}$ there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ we have $\tau \notin S$.

**Definition**

A set $G$ is $n$-generic relative to the perfect tree $T$ if $G$ is a path through $T$ and for every $\Sigma_n(T)$ subset $S$ of $T$, there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ with $\tau \in T$ we have $\tau \notin S$.

# 2-generic relative to some perfect tree

We wish to generalize this proof to come up with a negative result on a broad class of sets.

**Definition**

A set $G$ is $n$-generic if for every $\Sigma_n$ subset $S$ of $2^{<\omega}$ there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ we have $\tau \notin S$.

**Definition**

A set $G$ is $n$-generic relative to the perfect tree $T$ if $G$ is a path through $T$ and for every $\Sigma_n(T)$ subset $S$ of $T$, there is an $l$ such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ with $\tau \in T$ we have $\tau \notin S$.

**Definition**

A set $G$ is $n$-generic relative to some perfect tree if there exists a perfect tree $T$ such that $G$ is $n$-generic relative to $T$.

# 2-generic relative to some perfect tree (continued)

We can now use this to limit degrees of categoricity to a small, easily defined class.

**Theorem (Anderson)**

*For every $n$, there are only countably many sets that are not $n$-generic relative to any perfect tree.*

# 2-generic relative to some perfect tree (continued)

We can now use this to limit degrees of categoricity to a small, easily defined class.

**Theorem (Anderson)**

*For every $n$, there are only countably many sets that are not $n$-generic relative to any perfect tree.*

Generalizing the methods used to construct a degree below $\mathbf{0}''$ we can prove:

**Theorem (Anderson and Csima)**

*Let $G$ be 2-generic relative to some perfect tree and $\mathbf{g} = Deg(G)$. Then $\mathbf{g}$ is not a degree of categoricity.*

The theorem allows us to find a degree that is not a degree of categoricity between any set and its double jump.

**Corollary**

*Let $X$ and $A$ be sets such that $X$ is 2-generic ($A$). Then $\mathbf{x} \oplus \mathbf{a}$ is not a degree of categoricity.*

**Corollary**

*For every $\mathbf{x}$ there is a $\mathbf{y}$ such that $\mathbf{x} \leq_T \mathbf{y} \leq_T \mathbf{x}''$ and $\mathbf{y}$ is not a degree of categoricity.*

# Hyperimmune-free

We can also exclude degrees of categoricity from another class.

**Definition**

A degree $\mathbf{x}$ is hyperimmune-free if for every function $f \leq_T \mathbf{x}$ there is a computable function $g$ which dominates $f$.

We notice that all known degrees of categoricity are between jumps and hence hyperimmune.

# Hyperimmune-free

We can also exclude degrees of categoricity from another class.

**Definition**

A degree $\mathbf{x}$ is hyperimmune-free if for every function $f \leq_T \mathbf{x}$ there is a computable function $g$ which dominates $f$.

We notice that all known degrees of categoricity are between jumps and hence hyperimmune.

**Theorem (Anderson and Csima)**

*Let $\mathbf{x}$ be a noncomputable hyperimmune-free degree. Then $\mathbf{x}$ is not a degree of categoricity.*

# $\Sigma_2$ Degree

There are no hyperimmune-free degrees or degrees of sets 2-generic relative to some perfect tree that are $\Sigma_2$.

However, we can construct a $\Sigma_2$ set whose degree is not a degree of categoricity directly.

**Theorem (Anderson and Csima)**

*There is a $\Sigma_2$ degree that is not a degree of categoricity.*

# $\Sigma_2$ Degree (continued)

## Ideas for proof

- We construct $X$ to be c.e. in a $\emptyset'$ oracle.

# $\Sigma_2$ Degree (continued)

## Ideas for proof

- We construct $X$ to be c.e. in a $\emptyset'$ oracle.
- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.

# $\Sigma_2$ Degree (continued)

## Ideas for proof

- We construct $X$ to be c.e. in a $\emptyset'$ oracle.
- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.
- We weaken the requirement that $\mathbf{x} \in \mathrm{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \mathrm{CatSpec}(\mathcal{A}_m)$.
- Instead, for each $m \in \omega$ we construct a $Y_m \not\geq_T X$ such that for all $k$, if $X$ computes an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ then so does $Y_m$.

# $\Sigma_2$ Degree (continued)

## Ideas for proof

- We construct $X$ to be c.e. in a $\emptyset'$ oracle.
- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.
- We weaken the requirement that $\mathbf{x} \in \text{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \text{CatSpec}(\mathcal{A}_m)$.
- Instead, for each $m \in \omega$ we construct a $Y_m \not\geq_T X$ such that for all $k$, if $X$ computes an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ then so does $Y_m$.
- Each $Y_m$ witnesses $\mathbf{x}$ is not the least degree in $\text{CatSpec}(\mathcal{A}_m)$.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- We split each $Y_m$ into columns, $Y_m^{[l,k]}$.
- We maintain $Y_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all $t$.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- We split each $Y_m$ into columns, $Y_m^{[l,k]}$.
- We maintain $Y_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all $t$.
- If we appear unable to block $\Phi_l^X$ from becoming an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, we will try to make $f = \Phi_l^{Y_m^{[l,k]}}$ an isomorphism.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- We split each $Y_m$ into columns, $Y_m^{[l,k]}$.
- We maintain $Y_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all $t$.
- If we appear unable to block $\Phi_l^X$ from becoming an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, we will try to make $f = \Phi_l^{Y_m^{[l,k]}}$ an isomorphism.
- We build $X$ by finite extensions except at special stages called slides.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- Given $\sigma$ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is not a partial injective homomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$.
- At this point we have [roughly speaking] $X \restriction \sigma = Y_m^{[l,k]} \restriction \sigma$.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- Given $\sigma$ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is not a partial injective homomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$.
- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_m^{[l,k]} \upharpoonright \sigma$.
- If yes, we extend to $\tau$ and are done for $(l, m, k)$.
- If no, then for all $\gamma \supseteq \sigma$ we have $\Phi_l^\gamma$ is a partial injective homomorphism.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (continued)

- Given $\sigma$ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that $\Phi_l^\tau$ is not a partial injective homomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$.
- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_m^{[l,k]} \upharpoonright \sigma$.
- If yes, we extend to $\tau$ and are done for $(l, m, k)$.
- If no, then for all $\gamma \supseteq \sigma$ we have $\Phi_l^\gamma$ is a partial injective homomorphism.
- We attempt to build $Y_m^{[l,k]} \supseteq \sigma$ by finite extensions to ensure every $d \in \omega$ is in the domain and range of $f = \Phi_l^{Y_m^{[l,k]}}$.

# $\Sigma_2$ Degree (continued)

## Ideas for proof (conclusion)

- Problem: What if no extension for $Y_m^{[l,k]}$ puts $d$ into the domain and range of $f$?

# $\Sigma_2$ Degree (continued)

## Ideas for proof (conclusion)

- Problem: What if no extension for $Y_m^{[l,k]}$ puts $d$ into the domain and range of $f$?
- In this case we perform a slide. We change $X(t)$ from 0 to 1 for all $t$ where $X$ differs from $Y_m^{[l,k]}$.
- We now have $X = Y_m^{[l,k]}$ and since $\Phi_l^X$ cannot be made into an isomorphism, we are done for $(l, m, k)$.

## Ideas for proof (conclusion)

- Problem: What if no extension for $Y_m^{[l,k]}$ puts $d$ into the domain and range of $f$?
- In this case we perform a slide. We change $X(t)$ from 0 to 1 for all $t$ where $X$ differs from $Y_m^{[l,k]}$.
- We now have $X = Y_m^{[l,k]}$ and since $\Phi_l^X$ cannot be made into an isomorphism, we are done for $(l, m, k)$.
- Many weaker priorities are injured, but a finite injury construction is possible.

# Conclusion

There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

# Conclusion

There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

## Open questions

1. Is every 3-c.e. degree a degree of categoricity?

2. Is there a $\Delta_2$ degree which is not a degree of categoricity?

There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

**Open questions**

1. Is every 3-c.e. degree a degree of categoricity?

2. Is there a $\Delta_2$ degree which is not a degree of categoricity?

3. Is there a degree of categoricity which is not a strong degree of categoricity?

# Conclusion

There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

**Open questions**

1. Is every 3-c.e. degree a degree of categoricity?

2. Is there a $\Delta_2$ degree which is not a degree of categoricity?

3. Is there a degree of categoricity which is not a strong degree of categoricity?

Thank you.