Algorithms and implementations

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What is an algorithm?

- Basic aim: to “define” (or represent) algorithms in set theory, in the same way that we represent real numbers (Cantor, Dedekind) and random variables (Kolmogorov) by set-theoretic objects.

- What set-theoretic objects represent algorithms?

- When do two two set-theoretic objects represent the same algorithm? (The algorithm identity problem)

- In what way are algorithms effective?

- ...and do it so that the basic results about algorithms can be established rigorously (and naturally)

- ...and there should be some applications!
Plan for the lectures

**Lecture 1.** *Algorithms and implementations*
Discuss the problem and some ideas for solving it

**Lecture 2.** *English as a programming language*
Applications to Philosophy of language (and linguistics?)
synonymy and faithful translation $\sim$ algorithm identity

**Lecture 3.** *The axiomatic derivation of absolute lower bounds*
Applications to complexity (joint work with Lou van den Dries)
Do not depend on pinning down algorithm identity

Lectures 2 and 3 are independent of each other and mostly independent of Lecture 1

I will oversimplify, but: All lies are white (John Steel)
Outline of Lecture 1

Slogan: *The theory of algorithms is the theory of recursive equations*

(1) Three examples
(2) Machines vs. recursive definitions
(3) Recursors
(4) Elementary algorithms
(5) Implementations

Notation:

\[ \mathbb{N} = \{0, 1, 2, \ldots\} \]
\[ a \geq b \geq 1, \quad a = bq + r, \quad 0 \leq r < b \]
\[ \implies q = \text{iq}(a, b), \quad r = \text{rem}(a, b) \]
\[ \gcd(a, b) = \text{the greatest common divisor of } a \text{ and } b \]
\[ a \perp b \iff \text{rem}(a, b) = 1 \quad (a \text{ and } b \text{ are coprime}) \]
The Euclidean algorithm \( \varepsilon \)

For \( a, b \in \mathbb{N}, a \geq b \geq 1 \),

\[
\varepsilon : \ \gcd(a, b) = \text{if } (\text{rem}(a, b) = 0) \text{ then } b \text{ else } \gcd(b, \text{rem}(a, b))
\]

\( c_\varepsilon(a, b) \) = the number of divisions needed to compute \( \gcd(a, b) \) using \( \varepsilon \)

Complexity of the Euclidean

If \( a \geq b \geq 2 \), then \( c_\varepsilon(a, b) \leq 2 \log_2(a) \)

Proofs of the correctness and the upper bound are by induction on \( \max(a, b) \)
What is the Euclidean algorithm?

\[ \varepsilon : \text{gcd}(a, b) = \text{if } (\text{rem}(a, b) = 0) \text{ then } b \text{ else } \text{gcd}(b, \text{rem}(a, b)) \]

- It is an algorithm on \( \mathbb{N} \), from (relative to) the remainder function \( \text{rem} \) and it computes \( \text{gcd} : \mathbb{N}^2 \rightarrow \mathbb{N} \).
- It is needed to make precise the optimality of the Euclidean:

**Basic Conjecture**

*For every algorithm \( \alpha \) which computes on \( \mathbb{N} \) from \( \text{rem} \) the greatest common divisor function, there is a constant \( r > 0 \) such that for infinitely many pairs \( a \geq b \geq 1 \),

\[ c_{\alpha}(a, b) \geq r \log_2(a) \]
Sorting (alphabetizing)

Given an ordering $\leq$ on a set $A$ and any $u = \langle u_0, \ldots, u_{n-1} \rangle \in A^n$

$$\text{sort}(u) = \text{the unique, sorted (non-decreasing) rearrangement}$$
$$\nu = \langle u_{\pi(0)}, u_{\pi(1)}, \ldots, u_{\pi(n-1)} \rangle$$

where $\pi : \{0, \ldots, n-1\} \mapsto \{0, \ldots, n-1\}$ is a permutation

$$\text{head}(\langle u_0, \ldots, u_{n-1} \rangle) = u_0$$
$$\text{tail}(\langle u_0, \ldots, u_{n-1} \rangle) = \langle u_1, \ldots, u_{n-1} \rangle$$
$$\langle x \rangle \ast \langle u_0, \ldots, u_{n-1} \rangle = \langle x, u_0, \ldots, u_{n-1} \rangle \quad \text{(prepend)}$$
$$|\langle u_0, \ldots, u_{n-1} \rangle| = n \quad \text{(the length of u)}$$

$$h_1(u) = \text{the first half of u} \quad \text{(the first half)}$$
$$h_2(u) = \text{the second half of u} \quad \text{(the second half)}$$
The mergesort algorithm $\sigma_m$

$$\text{sort}(u) = \begin{cases} u & \text{if } |u| \leq 1, \\ \text{merge(sort}(h_1(u)), \text{sort}(h_2(u))) & \text{else} \end{cases}$$

$$\text{merge}(v, w) = \begin{cases} w & \text{if } |v| = 0, \\ v & \text{else, if } |w| = 0, \\ \langle v_0 \rangle \ast \text{merge}(\text{tail}(v), w) & \text{else, if } v_0 \leq w_0, \\ \langle w_0 \rangle \ast \text{merge}(v, \text{tail}(w)) & \text{otherwise.} \end{cases}$$

(1) If $v, w$ are sorted, then $\text{merge}(v, w) = \text{sort}(w \ast v)$

(2) The sorting and merging function satisfy these equations

(3) $\text{merge}(v, w)$ can be computed using no more than $|v| + |w| - 1$ comparisons

(4) $\text{sort}(u)$ can be computed by $\sigma_m$ using no more than $|u| \log_2(|u|)$ comparisons ($|u| > 1$)
What is the mergesort algorithm?

\[ \text{sort}(u) = \begin{cases} u & \text{if } |u| \leq 1, \\ \text{merge}(\text{sort}(h_1(u)), \text{sort}(h_2(u))) & \text{else}. \end{cases} \]

\[ \text{merge}(v, w) = \begin{cases} w & \text{if } |v| = 0, \\ v & \text{else, if } |w| = 0, \\ \langle v_0 \rangle \ast \text{merge}(\text{tail}(v), w) & \text{else, if } v_0 \leq w_0, \\ \langle w_0 \rangle \ast \text{merge}(v, \text{tail}(w)) & \text{otherwise}. \end{cases} \]

\[ c_{\sigma_m}(u) = \text{the number of comparisons needed to compute } \text{sort}(u) \text{ using } \sigma_m \leq |u| \log_2(|u|) \quad (|u| > 0) \]

- It is an algorithm from the ordering \( \leq \) and the functions head\((u)\), tail\((u)\), \(|u|\), \ldots

- It is needed to make precise the optimality of \( \sigma_m \):
  
  For every sorting algorithm \( \sigma \) from \( \leq \), head, tail, \ldots, there is an \( r > 0 \) and infinitely many sequences \( u \) such that

  \[ c_\sigma(u) \geq r|u| \log_2(|u|) \quad \text{(well known)} \]
The Gentzen Cut Elimination algorithm

Every proof $d$ of the Gentzen system for Predicate Logic can be transformed into a cut-free proof $\gamma(d)$ with the same conclusion

$$\gamma(d) = \text{if } T_1(d) \text{ then } f_1(d)$$

else if $T_2(d)$ then $f_2(\gamma(\tau(d)))$

else $f_3(\gamma(\sigma_1(d)), \gamma(\sigma_2(d)))$

- It is a recursive algorithm from natural syntactic primitives, very similar in logical structure to the mergesort
- **Main Fact:** $|\gamma(d)| \leq e(\rho(d), |d|)$, where $|d|$ is the length of the proof $d$, $\rho(d)$ is its cut-rank, and

$$e(0, k) = k, \quad e(n + 1, k) = 2^{e(n, k)}$$
The infinitary Gentzen algorithm

If we add the $\omega$-rule to the Gentzen system for Peano arithmetic, then cuts can again be eliminated by an extension of the finitary Gentzen algorithm

$$\gamma^*(d) = \begin{cases} f_1(d) & \text{if } T_1(d) \\ f_2(\gamma^*(\tau(d))) & \text{else if } T_2(d) \\ f_3(\gamma^*(\sigma_1(d)), \gamma^*(\sigma_2(d))) & \text{else if } T_3(d) \\ f_4(\lambda(n)\gamma^*(\rho(n,d))) & \text{else} \end{cases}$$

where $f_4$ is a functional embodying the $\omega$-rule

- Again $|\gamma^*(d)| \leq e(\rho(d), |d|)$, where cut-ranks and lengths of infinite proofs are ordinals, $e(\alpha, \beta)$ is defined by ordinal recursion, and so every provable sentence has a cut-free proof of length less than $\varepsilon_0 = \text{the least ordinal } > 0$ and closed under $\alpha \mapsto \omega^\alpha$
Abstract machines (computation models)

A machine \( m : X \rightsquigarrow Y \) is a tuple \( (S, \text{input}, \sigma, T, \text{output}) \) such that

1. \( S \) is a non-empty set (of states)
2. \( \text{input} : X \to S \) is the input function
3. \( \sigma : S \to S \) is the transition function
4. \( T \) is the set of terminal states, \( T \subseteq S \)
5. \( \text{output} : T \to Y \) is the output function

\[
\overline{m}(x) = \text{output}(\sigma^n(\text{input}(x)))
\]

where \( n = \text{least such that } \sigma^n(\text{input}(x)) \in T \)
Infinitary algorithms are not machines

- It is useful to think of the infinitary Gentzen “effective procedure” as an algorithm
- There are applications of infinitary algorithms (in Lecture 2)
- Machines are special algorithms which implement finitary algorithms
- The relation between an (implementable) algorithm and its implementations is interesting
Which machine is the Euclidean?

\[
\varepsilon : \text{gcd}(a, b) = \text{if } (\text{rem}(a, b) = 0) \text{ then } b \text{ else } \text{gcd}(b, \text{rem}(a, b))
\]

- Must specify a set of states, an input function, a transition function, etc.
- This can be done, in many ways, generally called implementations of the Euclidean
- The choice of a “natural” (abstract) implementation is irrelevant for the correctness and the log upper bound of the Euclidean, which are derived directly from the recursive equation above and apply to all implementations
- Claim: \( \varepsilon \) is completely specified by the equation above
Which machine is the mergesort algorithm?

sort(u) = if (|u| ≤ 1) then u else merge(sort(h1(u)), sort(h2(u)))

merge(v, w) = \[
\begin{align*}
& w & \text{if } |v| = 0, \\
& v & \text{else, if } |w| = 0, \\
& \langle v_0 \rangle \ast \text{merge(tail}(v), w) & \text{else, if } v_0 \leq w_0, \\
& \langle w_0 \rangle \ast \text{merge}(v, \text{tail}(w)) & \text{otherwise.}
\end{align*}
\]

- Many (essentially) different implementations sequential (with specified orders of evaluation), parallel, ...
- The correctness and $n \log_2(n)$ upper bound are derived directly from a (specific reading) of these recursive equations
- They should apply to all implementations of the mergesort
- Claim: $\sigma_m$ is completely specified by the system above
- Task: Define $\sigma_m$, define implementations, prove
Slogans and questions

- Algorithms compute functions \textit{from} specific primitives
- They are specified by \textit{systems of recursive equations}
- An algorithm \textit{is} (faithfully modeled by) \textit{the semantic content} of a system of recursive equations
- Machines are algorithms, but not all algorithms are machines
- Some algorithms have \textit{machine implementations}
- An algorithm \textit{codes} all its implementation-independent properties
- What is the relation between an algorithm and its implementations? … or between two implementations of the same algorithm?

Main slogan

\textit{The theory of algorithms is the theory of recursive equations}

(Skip \textit{non-deterministic} algorithms and \textit{fairness})
Monotone recursive equations

- A complete poset is a partial ordered set \( D = (\text{Field}(D), \leq_D) \) in which every directed set has a least upper bound.

- Standard example: 
  \((X \rightarrow Y) = \text{the set of all partial functions } f : X \rightarrow Y\)

- A function \( f : D \rightarrow E \) is monotone if \( x \leq_D y \implies f(x) \leq_E f(y) \)
  \( (f : X \rightarrow Y \text{ is a monotone function on } X \text{ to } Y \cup \{\bot\}) \)

- For every monotone \( f : D \rightarrow D \) on a complete \( D \), the equation \( x = f(x) \) has a least solution.

- Complete posets (domains) are the basic objects studied in Scott’s Denotational Semantics for programming languages.

- Much of this work can be viewed as a refinement of Denotational Semantics (which interprets programs by algorithms).
A **recursor** \( \alpha : X \rightsquigarrow W \) is a tuple \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k) \) such that

1. \( X \) is a poset, \( W \) is a complete poset
2. \( D_1, \ldots, D_k \) are complete posets, \( D_\alpha = D_1 \times \cdots \times D_k \), the solution space of \( \alpha \)
3. \( \alpha_i : X \times D_\alpha \to D_i \) is monotone \((i = 1, \ldots, k)\)
4. \( \tau_\alpha(x, \vec{d}) = (\alpha_1(x, \vec{d}), \ldots, \alpha_k(x, \vec{d})) \) is the transition function, \( \tau_\alpha : X \times D_\alpha \to D_\alpha \)
5. \( \alpha_0 : X \times D_1 \times \cdots \times D_k \to W \) is monotone, the output map \( \bar{\alpha}(x) = \alpha_0(x, \vec{d}_1, \ldots, \vec{d}_k) \) for the least solution of \( \vec{d} = \tau_\alpha(x, \vec{d}) \)

We write \( \alpha(x) = \alpha_0(x, \vec{d}) \) where \( \{\vec{d} = \tau_\alpha(x, \vec{d})\} \)
Recursor isomorphism

Two recursors

\[ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k), \quad \alpha' = (\alpha'_0, \alpha'_1, \ldots, \alpha'_m) : X \leadsto W \]

are isomorphic \((\alpha \simeq \alpha')\) if

1. \(k = m\) (same number of parts)
2. There is a permutation \(\pi : \{1, \ldots, k\}\) and poset isomorphisms \(\rho_i : D_i \rightarrow D'_{\pi(i)}\) \((i = 1, \ldots, k)\) such that...

   (the order of the equations in the system \(\vec{d} = \tau_\alpha(x, \vec{d})\) does not matter)

Isomorphic recursors \(\alpha, \alpha' : X \leadsto W\) compute the same function

\[ \overline{\alpha} = \overline{\alpha'} : X \rightarrow W \]
Machines or recursors?

With each machine \( m = (S, \text{input}, \sigma, T, \text{output}) : X \xrightarrow{\sim} Y \) we associate the tail recursor

\[
\tau_m(x) = p(\text{input}(x)) \quad \text{where} \quad \{ p = \lambda(s)[\text{if } (s \in T) \text{ then output}(s) \text{ else } p(\sigma(s))] \}
\]

- \( m \) and \( \tau_m \) compute the same partial function \( \tau_m = \overline{m} : X \rightarrow Y \)

- **Theorem** (with V. Paschalis) The map \( m \mapsto \tau_m \) respects isomorphisms, \( m \sim m' \iff \tau_m \sim \tau_m' \)

- The question is one of choice of terminology (because the mergesort system is also needed)

- Yuri Gurevich has argued that algorithms are machines (and of a very specific kind)

- Jean-Yves Girard has also given similar arguments
Elementary (first order) algorithms

Algorithms which compute partial functions from given partial functions

(Partial, pointed) algebra \( \mathbf{M} = (M, 0, 1, \Phi^M) \)

where \( 0, 1 \in M, \Phi \) is a set of function symbols (the vocabulary)
and \( \Phi^M = \{ \phi^M \}_{\phi \in \Phi} \), with \( \phi^M : M^n \phi \to M \) for each \( \phi \in \Phi \)

\( \mathbf{N}_\epsilon = (\mathbb{N}, 0, 1, \text{rem}) \), the Euclidean algebra
\( \mathbf{N}_u = (\mathbb{N}, 0, 1, S, \text{Pd}) \), the \textit{unary numbers}
\( \mathbf{N}_b = (\mathbb{N}, 0, 1, \text{Parity}, \text{iq}_2, (x \mapsto 2x), (x \mapsto 2x + 1)) \), the \textit{binary numbers}
\( \mathbf{A}^* = (\mathbb{A}^*, 0, 1, \leq, \text{head}, \text{tail}, \ldots) \), the mergesort algebra, with \( 0, 1 \in \mathbb{A}^* \)

Standard model-theoretic notions must be mildly adapted, for example for (partial) \textbf{subalgebras}:

\[ \mathbf{U} \subseteq_p \mathbf{M} \iff \{0, 1\} \subseteq U \subseteq M \text{ and for all } \phi, \phi^U \subseteq \phi^M \]
Recursive (McCarthy) programs of $\mathbf{M} = (\mathcal{M}, 0, 1, \Phi^\mathbf{M})$

Explicit $\Phi$-terms (with partial function variables and conditionals)

$$A : \equiv 0 \mid 1 \mid v_i \mid \phi(A_1, \ldots, A_n) \mid p_i^n(A_1, \ldots, A_n) \mid \text{if } (A = 0) \text{ then } B \text{ else } C$$

Recursive program (only $\vec{x}_i, p_1, \ldots, p_K$ occur in each part $A_i$):

$$A : \begin{cases} p_A(\vec{x}_0) = A_0 \\ p_1(\vec{x}_1) = A_1 \\ \vdots \\ p_K(\vec{x}_K) = A_K \end{cases} \quad (A_0 : \text{the head}, (A_1, \ldots, A_K) : \text{the body})$$

- What is the semantic content of the system $A$?
The recursor of a program in $\mathbb{M}$

\[ A : \left\{ \begin{array}{l}
  p_A(\vec{x}_0) = A_0 \\
p_1(\vec{x}_1) = A_1 \\
  \vdots \\
p_K(\vec{x}_K) = A_K
\end{array} \right. \]

\[ \tau(A, \mathbb{M})(\vec{x}) = \text{den}(A_0, \mathbb{M})(\vec{x}, \vec{p}) \text{ where} \]

\[ \left\{ p_1 = \lambda(\vec{x}_1)\text{den}(A_1, \mathbb{M})(\vec{x}_1, \vec{p}), \ldots, p_K = \lambda(\vec{x}_K)\text{den}(A_K, \mathbb{M})(\vec{x}_K, \vec{p}) \right\} \]

$\tau(A, \mathbb{M})$ is not exactly the algorithm expressed by $A$ in $\mathbb{M}$.

For example, if $A : p_A(\vec{x}) = A_0(\vec{x})$ has empty body, then

\[ \tau(A, \mathbb{M})(\vec{x}) = \text{den}(A_0, \mathbb{M})(\vec{x}) \text{ where} \left\{ \right\} \]

is just the function defined on $\mathbb{M}$ by $A_0$

(which may involve much explicit computation)
The problem of defining implementations

van Emde Boas:

*Intuitively, a simulation of [one class of computation models] $M$ by [another] $M'$ is some construction which shows that everything a machine $M_i \in M$ can do on inputs $x$ can be performed by some machine $M'_i \in M'$ on the same inputs as well;*

We will define a *reducibility relation* $\alpha \leq_r \beta$ and call a machine $m$ an *implementation* of $\alpha$ if $\alpha \leq_r \tau_m$

(where $\tau_m$ is the recursor representation of the machine $m$)
Recursor reducibility

Suppose $\alpha, \beta : X \leadsto W$, (e.g., $\beta = r_m$ where $m : X \leadsto W$):
A *reduction* of $\alpha$ to $\beta$ is any monotone mapping

$$\pi : X \times D_\alpha \rightarrow D_\beta$$

such that the following three conditions hold, for every $x \in X$ and every $d \in D_\alpha$:

1. (R1) $\tau_\beta(x, \pi(x, d)) \leq \pi(x, \tau_\alpha(x, d))$.
2. (R2) $\beta_0(x, \pi(x, d)) \leq \alpha_0(x, d)$.
3. (R3) $\overline{\alpha}(x) = \overline{\beta}(x)$.

$\alpha \leq_r \beta$ if a reduction exists

$m$ implements $\alpha$ if $\alpha \leq_r r_m$
Theorem (with Paschalis)

For any recursive program $A$ in an algebra $M$, the standard implementation of $A$ is an implementation of $\tau(A, M)$

... Uniformly enough, so that (with the full definitions), the standard implementation of $A$ implements the elementary algorithm expressed by $A$ in $M$

... And this is true of all familiar implementations of recursive programs

... so that the basic (complexity and resource use) upper and lower bounds established from the program $A$ hold of all implementations of $A$

And for the applications to complexity theory, we work directly with the recursive equations of $A$
English as a programming language

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Tarski Lecture 2, March 5, 2008
Frege on sense

“[the sense of a sign] may be the common property of many people”

Meanings are public (abstract?) objects

“The sense of a proper name is grasped by everyone who is sufficiently familiar with the language . . . Comprehensive knowledge of the thing denoted . . . we never attain”

Speakers of the language know the meanings of terms

“The same sense has different expressions in different languages or even in the same language”

“The difference between a translation and the original text should properly not overstep the [level of the idea]”

Faithful translation should preserve meaning
Outline of Lecture 2

Slogan:

The meaning of a term is the algorithm which computes its denotation

(1) Formal Fregean semantics in $L^\lambda_r(K)$
(2) Meaning and synonymy in $L^\lambda_r(K)$
(3) What are the objects of belief? (Local synonymy)
(4) The decision problem for synonymy

Sense and denotation as algorithm and value (1994)
A logical calculus of meaning and synonymy (2006)
Two aspects of situated meaning (with E. Kalyvianaki, to appear)
Posted in www.math.ucla.edu/~ynm
The methodology of formal Fregean semantics

- An *interpreted formal language* $L$ is selected
- The *rendering* operation on a fragment of English:
  
  \[
  \text{English expression} + \text{informal context} \xrightarrow{\text{render}} \text{formal expression} + \text{state}
  \]
  
- Semantic values (denotations, meanings, etc.) are defined rigorously for the formal expressions of $L$ and assigned to English expressions via the rendering operation

- Montague: $L$ *should be a higher type language*  
  (to interpret co-ordination, co-indexing, . . . )

- Claim: $L$ *should be a programming language*  
  (to interpret self-reference and to define meanings properly)
The typed $\lambda$-calculus with recursion $L^\lambda_r(K)$ - types

An extension of the typed $\lambda$-calculus, into which Montague’s Language of Intensional Logic LIL can be easily interpreted (Gallin)

Basic types $b \equiv e \mid t \mid s$ (entities, truth values, states)

Types: $\sigma :\equiv b \mid (\sigma_1 \rightarrow \sigma_2)$

Abbreviation: $\sigma_1 \times \sigma_2 \rightarrow \tau \equiv (\sigma_1 \rightarrow (\sigma_2 \rightarrow \tau))$

Every non-basic type is uniquely of the form

$$\sigma \equiv \sigma_1 \times \cdots \times \sigma_n \rightarrow b$$

level$(b) = 0$

level$(\sigma_1 \times \cdots \times \sigma_n \rightarrow b) = \max\{\text{level}(\sigma_1), \ldots, \text{level}(\sigma_n)\} + 1$
The typed $\lambda$-calculus with recursion $L^\lambda_r(K)$ - syntax

**Pure variables:** $v_0^\sigma, v_1^\sigma, \ldots$, for each type $\sigma$ ($v : \sigma$)

**Pure parameters:** $\bar{u}$ for each state $u$ (for convenience only)

**Recursive variables:** $p_0^\sigma, p_1^\sigma, \ldots$, for each type $\sigma$ ($p : \sigma$)

**Constants:** A finite set $K$ of typed constants (run, cow, he, the, every)

**Terms** – with assumed type restrictions and assigned types ($A : \sigma$)

$$A : \equiv v \mid \bar{u} \mid p \mid c \mid B(C) \mid \lambda(v)(B)$$

$$\mid A_0 \text{ where } \{p_1 = A_1, \ldots, p_n = A_n\}$$

$$C : \sigma, B : (\sigma \rightarrow \tau) \implies B(C) : \tau$$

$$v : \sigma, B : \tau \implies \lambda(v)(B) : (\sigma \rightarrow \tau)$$

$$A_0 : \sigma \implies A_0 \text{ where } \{p_1 = A_1, \ldots, p_n = A_n\} : \sigma$$

**Abbreviation:** $A(B, C, D) \equiv A(B)(C)(D)$
\( L^\lambda_r(K) \) - denotational semantics

- We are given basic sets \( \mathbb{T}_s, \mathbb{T}_e \) and \( \mathbb{T}_t \subseteq \mathbb{T}_e \) for the basic types

\[
\mathbb{T}_{\sigma \rightarrow \tau} = \text{the set of all functions } f : \mathbb{T}_\sigma \rightarrow \mathbb{T}_\tau
\]

\[
P_b = \mathbb{T}_b \cup \{ \bot \} = \text{the “flat poset” of } \mathbb{T}_b
\]

\[
P_{\sigma \rightarrow \tau} = \text{the set of all functions } f : \mathbb{T}_\sigma \rightarrow P_\tau
\]

- We are given an object \( c : P_\sigma \) for each constant \( c : \sigma \)

- Pure variables of type \( \sigma \) vary over \( \mathbb{T}_\sigma \); recursive ones over \( P_\sigma \)

- If \( A : \sigma \) and \( \pi \) is a type-respecting assignment to the variables, then \( \text{den}(A)(\pi) \in P_\sigma \)

- Recursive terms are interpreted by the taking of least-fixed-points
Rendering natural language in $L^\lambda_r(K)$

$\tilde{t} \equiv (s \to t)$ (type of Carnap intensions)

$\tilde{e} \equiv (s \to e)$ (type of individual concepts)

Abelard loves Eloise $\xrightarrow{\text{render}}$ loves(Abelard,Eloise) : $\tilde{t}$

Bush is the president $\xrightarrow{\text{render}}$ eq(Bush,the(president)) : $\tilde{t}$

liar $\xrightarrow{\text{render}}$ p where $\{p = \neg p\}$ : t

truth-teller $\xrightarrow{\text{render}}$ p where $\{p = p\}$ : t

Abelard, Eloise, Bush : $\tilde{e}$

president : $\tilde{e} \to \tilde{t}$, eq : $\tilde{e} \times \tilde{e} \to \tilde{t}$

$\neg : t \to t$, the : ($\tilde{e} \to \tilde{t}$) $\to \tilde{e}$

$\text{den(liar)} = \text{den(truth-teller)} = \bot$
Co-ordination and co-indexing in $L^\lambda_r(K)$

John stumbled and fell vs. John stumbled and he fell

John stumbled and fell $\xrightarrow{\text{render}} \lambda(x)\left(\text{stumbled}(x) \& \text{fell}(x)\right)(\text{John})$

(predication after co-ordination)

This is in Montague’s LIL (as it is interpreted in $L^\lambda_r(K)$)

John stumbled and he fell $\xrightarrow{\text{render}} \text{stumbled}(j) \& \text{fell}(j)$ where $\{j = \text{John}\}$

(conjunction after co-indexing)

The logical form of this sentence cannot be captured faithfully in LIL — recursion models co-indexing preserving logical form
Can we say nonsense in $L^\lambda_r(K)$?

Yes!
In particular, we have parameters over states—so we can explicitly refer to the state (even to two states in one term); LIL does not allow this, because we cannot do this in English.

Consider the terms

$$A \equiv \text{rapidly(tall)}(\text{John}), \quad B \equiv \text{rapidly(sleeping)}(\text{John}) : \tilde{t}$$

$A$ and $B$ are terms of LIL, not the renderings of correct English sentences.

- The target formal language is a tool for defining rigorously the desired semantic values and it needs to be richer than a direct formalization of the relevant fragment of English—to insure compositionality, if for no other reason.
Meaning and synonymy in $L^\lambda_r(K)$

For a sentence $A : \tilde{t}$, the Montague sense of $A$ is $\text{den}(A) : \mathbb{T}_s \rightarrow \mathbb{T}_t$, so that

there are infinitely many primes

is Montague-synonymous with $1 + 1 = 2$

In $L^\lambda_r(K)$: The meaning of a term $A$ is \textit{modeled by an algorithm} $\text{int}(A)$ which computes $\text{den}(A)(\pi)$ for every $\pi$

The \textit{referential intension} $\text{int}(A)$ is compositionally determined from $A$

$\text{int}(A)$ is an abstract (not necessarily implementable) recursive algorithm of $L^\lambda_r(K)$

Referential synonymy: $A \approx B \iff \text{int}(A) \sim \text{int}(A)$
Reduction, Canonical Forms and the Synonymy Theorem

- A reduction relation $A \Rightarrow B$ is defined on terms of $L^\lambda_r(K)$
- Each term $A$ is reducible to a unique (up to congruence) irreducible recursive term, its canonical form

$$A \Rightarrow \text{cf}(A) \equiv A_0 \text{ where } \{p_1 = A_1, \ldots, p_n = A_n\}$$

- $\text{int}(A) = (\text{den}(A_0), \text{den}(A_1), \ldots, \text{den}(A_n))$
- The parts $A_0, \ldots, A_n$ of $A$ are irreducible, explicit terms
- $\text{cf}(A)$ models the logical form of $A$
- **Synonymy Theorem.** $A \approx B$ if and only if

$$B \Rightarrow \text{cf}(B) \equiv B_0 \text{ where } \{p_1 = B_1, \ldots, p_m = B_m\}$$

so that $n = m$ and for $i \leq n$, $\text{den}(A_i) = \text{den}(B_i)$
Is this notion of meaning Fregean?

Evans (in a discussion of Dummett’s similar, computational interpretations of Frege’s sense):

“This leads [Dummett] to think generally that the sense of an expression is (not a way of thinking about its [denotation], but) a method or procedure for determining its denotation. So someone who grasps the sense of a sentence will be possessed of some method for determining the sentence’s truth value

...ideal verificationism

...there is scant evidence for attributing it to Frege”

Converse question: For a sentence $A$, if you possess the method determined by $A$ for determining its truth value, do you then “grasp” the sense of $A$?

(Sounds more like Davidson rather than Frege)
The reduction calculus

Bush is the president $\xrightarrow{\text{render}} \text{eq}(\text{Bush})(\text{the(president)})$

$\Rightarrow \text{eq}(\text{Bush})(L) \text{ where } \{ L = \text{the(president)} \}$

$\Rightarrow \text{eq}(\text{Bush})(L) \text{ where } \{ L = \text{the}(p) \text{ where } \{ p = \text{president} \} \}$

$\Rightarrow \text{eq}(\text{Bush})(L) \text{ where } \{ L = \text{the}(p), p = \text{president} \}$

$\Rightarrow \left( \text{eq}(b) \text{ where } \{ b = \text{Bush} \} \right)(L) \text{ where } \{ L = \text{the}(p), p = \text{president} \}$

$\Rightarrow \left( \text{eq}(b)(L) \text{ where } \{ b = \text{Bush} \} \right) \text{ where } \{ L = \text{the}(p), p = \text{president} \}$

$\Rightarrow_{\text{cf}} \text{eq}(b)(L) \text{ where } \{ b = \text{Bush}, L = \text{the}(p), p = \text{president} \}$

He is the president $\xrightarrow{\text{render}} \text{eq}(\text{He})(\text{the(president)})$

$\Rightarrow_{\text{cf}} \text{eq}(b)(L) \text{ where } \{ b = \text{He}, L = \text{the}(p), p = \text{president} \}$
The reduction calculus

John loves and honors his father

\[ \text{render} \quad \left( \lambda(x)(\text{loves}(j, x) \land \text{honors}(j, x)) \right)(\text{father}(j)) \text{ where } \{j = \text{John}\} \]

\[ \Rightarrow \left[ \left( \lambda(x)(\text{loves}(j, x) \land \text{honors}(j, x)) \right)(f) \text{ where } \{f = \text{father}(j)\} \right] \]

\[ \Rightarrow \left( \lambda(x)(\text{loves}(j, x) \land \text{honors}(j, x)) \right)(f) \]

\[ \text{ where } \{f = \text{father}(j), j = \text{John}\} \]

\[ \Rightarrow \left( \lambda(x) \left[ (l \land h) \text{ where } \{l = \text{loves}(j, x), h = \text{honors}(j, x)\} \right] \right)(f) \]

\[ \text{ where } \{f = \text{father}(j), j = \text{John}\} \]

\[ \Rightarrow \left( \lambda(x)(l(x) \land h(x)) \right) \]

\[ \text{ where } \{l = \lambda(x)\text{loves}(j, x), h = \lambda(x)\text{honors}(j, x)\} \]

\[ \Rightarrow \lambda(x)(l(x) \land h(x))(f) \]

\[ \text{ where } \{l = \text{loves}(j, \cdot), h = \text{honors}(j, \cdot), f = \text{father}(j), j = \text{John}\} \]
Utterances, local meanings, local synonymy

An utterance is a pair \((A, u)\), where \(A\) is a sentence, \(A : \tilde{t}\) and \(u\) is a state; it is expressed in \(L^\lambda_r(K)\) by the term \(A(\bar{u})\)

The local meaning of \(A\) at the state \(u\) is \(\text{int}(A(\bar{u}))\)

\[
A \approx_u B \iff A(\bar{u}) \approx B(\bar{u})
\]

Bush is the president\((\bar{u})\)
\[
\Rightarrow_{\text{cf}} \text{eq}(b)(L)(\bar{u}) \quad \text{where} \quad \{b = \text{Bush}, L = \text{the}(p), p = \text{president}\}
\]

He is the president\((\bar{u})\)
\[
\Rightarrow_{\text{cf}} \text{eq}(b)(L)(\bar{u}) \quad \text{where} \quad \{b = \text{He}, L = \text{the}(p), p = \text{president}\}
\]

Bush is the president \(\not\approx_u\) He is the president

even if at the state \(\bar{u}\), \(\text{He}(\bar{u}) = \text{Bush}(\bar{u})\)
Three aspects of meaning for a sentence $A : \tilde{\tau}$

Referential intension $\text{int}(A)$ Referential synonymy $\approx$
Local meaning at $u$ $\text{int}(A(\bar{u}))$ Local synonymy $\approx_u$
Factual content at $u$ $\text{FC}(A, u)$ Factual synonymy $\approx_{f,u}$

The *factual content* of a sentence at a state $u$ gives a *representation of the world* at $u$ (Eleni Kalyvianaki’s Ph.D. Thesis)

Bush is the president $\not\approx_u$ He is the president
Bush is the president $\approx_{f,u}$ He is the president

Claim: *The objects of belief are local meanings*

The distinction between local meaning and factual content are related to David Kaplan’s distinction between the *character* and *content* of a sentence at a state
Some referential (global) synonymies and non-synonymies

- There are infinitely many primes $\not\approx 1 + 1 = 2$
- $A & B \approx B & A$
- The morning star is the evening star $\approx$ The evening star is the morning star
  (This fails with Montague’s renderings)
- Abelard loves Eloise $\approx$ Eloise is loved by Abelard (Frege)
- $2 + 3 = 6 \approx 3 + 2 = 6$ (with + and the numbers primitive)
- liar $\not\approx$ truthteller
- John stumbled and he fell $\overset{\text{render}}{\iff}$ $A \equiv \text{stumbled}(j) & \text{fell}(j)$ where $\{j = \text{John}\}$
  $A$ is not $\approx$ with any explicit term (including any term from LIL)
Is referential synonymy decidable?

**Synonymy Theorem.** $A \approx B$ if and only if

$$A \Rightarrow \text{cf}(A) \equiv A_0 \text{ where } \{p_1 = A_1, \ldots, p_n = A_n\}$$

$$B \Rightarrow \text{cf}(B) \equiv B_0 \text{ where } \{p_1 = B_1, \ldots, p_n = B_n\}$$

so that for $i = 0, \ldots, n$ and all $\pi$, $\text{den}(A_i)(\pi) = \text{den}(B_i)(\pi)$.

- Synonymy is reduced to denotational equality for explicit, irreducible terms (the truth facts of $A$)
- Denotational equality for arbitrary terms is undecidable (there are constants, with fixed interpretations)
- The explicit, irreducible terms are very special — but by no means trivial!
The synonymy problem for $L_\lambda^\rho(K)$ (with finite $K$)

- The decision problem for $L_\lambda^\rho(K)$-synonymy is open

**Theorem** *If the set of constants $K$ is finite, then synonymy is decidable for terms of adjusted level $\leq 2$*

These include terms constructed “simply” from

Names of “pure” objects $\quad 0, 1, 2, \emptyset, \ldots : \varepsilon$

Names, demonstratives $\quad$ John, I, he, him : $\tilde{\varepsilon}$

Common nouns $\quad$ man, unicorn, temperature : $\varepsilon \rightarrow \tilde{t}$

Adjectives $\quad$ tall, young : $(\varepsilon \rightarrow \tilde{t}) \rightarrow (\varepsilon \rightarrow \tilde{t})$

Propositions $\quad$ it rains : $\tilde{t}$

Intransitive verbs $\quad$ stand, run, rise : $\varepsilon \rightarrow \tilde{t}$

Transitive verbs $\quad$ find, loves, be : $\varepsilon \times \varepsilon \rightarrow \tilde{t}$

Adverbs $\quad$ rapidly : $(\varepsilon \rightarrow \tilde{t}) \rightarrow (\varepsilon \rightarrow \tilde{t})$

Proof is by reducing this claim to the Main Theorem in the 1994 paper (for a corrected version see www.math.ucla.edu/~ynm)
Explicit, irreducible identities that must be known

- Los Angeles = LA  (Athens = Αθήνα)
- \( x \land y = y \land x \)
- \( \text{between}(x, y, z) = \text{between}(x, z, y) \)
- \( \text{love}(x, y) = \text{be_loved}(y, x) \)

A dictionary is needed—but what kind and how large?

\[
ev_2(\lambda(u_1, u_2)r(u_1, u_2, \vec{a}), b, z) = ev_1(\lambda(v)r(v, z, \vec{a}), b)
\]

Evaluation functions: both sides are equal to \( r(b, z, \vec{a}) \)

The dictionary line which determines this is (essentially)

\[
\lambda(s)x(s, z) = \lambda(s)y(s) \implies ev_2(x, b, z) = ev_1(y, b)
\]
The form of the decision algorithm

- A finite list of true dictionary lines is constructed, which codifies the relationships between the constants.
- Given two explicit, irreducible terms $A, B$ of adjusted level $\leq 2$, we construct (effectively) a finite set $L(A, B)$ of lines such that
  
  \[ \models A = B \]
  \[ \iff \] every line in $L(A, B)$ is congruent to one in the dictionary

- It is a lookup algorithm, justified by a finite basis theorem.
- Complexity: NP; the graph isomorphism problem is reducible to the synonymy problem for very simple (propositional) recursive terms.
The axiomatic derivation of absolute lower bounds

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Tarski Lecture 3, March 7, 2008
A lower bound result

**Theorem** (van den Dries, ynmd)

If an algorithm $\alpha$ decides the coprimeness relation $x \perp y$ on $\mathbb{N}$ from the primitives $\leq, +, \div, \text{iq}, \text{rem}$, then for infinitely many $a, b$

$$c^{s}_{\alpha}(a, b) > \frac{1}{10} \log \log (\max(a, b))$$

(\*)

In fact (\*) holds for all solutions of Pell’s equation, $a^2 = 1 + 2b^2$

- $\text{iq}(x, y), \text{rem}(x, y)$ are the integer quotient and remainder
- $c^{s}_{\alpha}(x, y)$ counts the number of applications of the primitives in the computation
- **Claim:** This applies to all algorithms from the specified primitives
- The Euclidean decides coprimeness from rem with complexity

$$c^{s}_{\epsilon}(a, b) \leq 2 \log (\min(a, b)) \quad (\min(a, b) \geq 2)$$
Outline of Lecture 3

Slogan: *Lower bound results are the undecidability facts about decidable problems*

...and so they should be (to some extent) a matter of logic

(1) Tweak logic (a bit) so it applies smoothly to computation theory
(2) Three (simple) axioms for elementary algorithms, a la *abstract model theory*
(3) Lower bounds from the axioms
(4) Lower bounds for elementary algorithms on logical extensions

*Is the Euclidean algorithm optimal among its peers?* (with vDD, 2004)
*Arithmetic complexity* (with vDD, to appear)
Partial algebras, embeddings and subalgebras

- A (Partial, pointed) algebra is a structure $\mathbf{M} = (M, 0, 1, \Phi^M)$ where $0, 1 \in M$, $\Phi$ is a set of function symbols (the vocabulary) and $\Phi^M = \{ \phi^M \}_{\phi \in \Phi}$, with $\phi^M : M^n \phi \to M$ for each $\phi \in \Phi$.

- An embedding $\iota : U \hookrightarrow M$ from one $\Phi$-algebra into another is any injection $\iota : U \hookrightarrow M$ such that
  
  $\iota(0^U) = 0^M$, \hspace{0.5cm} $\iota(1^U) = 1^M$,

  and for all $\phi \in \Phi, x_1, \ldots, x_n, w \in U$,

  $\phi^U(x_1, \ldots, x_n) = w \implies \phi^M(\iota x_1, \ldots, \iota x_n) = \iota w$.

- $U \subseteq_p \mathbf{M}$ if the identity $I : U \hookrightarrow M$ is an embedding.
Algebra restrictions

\( \mathbb{N}_\varepsilon = (\mathbb{N}, 0, 1, \text{rem}) \), the Euclidean algebra
\( \mathbb{N}_u = (\mathbb{N}, 0, 1, S, \text{Pd}) \), the \textit{unary numbers}
\( \mathbb{N}_b = (\mathbb{N}, 0, 1, \text{Parity}, \text{iq}_2, (x \mapsto 2x), (x \mapsto 2x + 1)) \), the \textit{binary numbers}

For \( \mathbb{M} = (M, 0, 1, \Phi^M) \) and \( \{0, 1\} \subseteq U \subseteq M \), let

\[ \mathbb{M} \upharpoonright U = (U, 0, 1, \Phi^U), \]

where for \( \phi \in \Phi \),

\[ \phi^U (\vec{x}) = w \iff \vec{x}, w \in U \land \phi^M (\vec{x}) = w \]

For finite \( U \subset \mathbb{N} \), \( \mathbb{N}_u \upharpoonright U \) is a finite, properly partial subalgebra of \( \mathbb{N} \)
Subalgebras generated from the input, $\mathbf{M} = (M, 0, 1, \Phi^M)$

For $\vec{x} = x_1, \ldots, x_n \in M$, set

$$G_0(\vec{x}) = \{0, 1, x_1, \ldots, x_n\}$$

$$G_{m+1}(\vec{x}) = G_m(\vec{x}) \cup \{\phi^M(\vec{u}) \mid \phi \in \Phi, \vec{u} \in G_m(\vec{x}) \text{ and } \phi^M(\vec{u}) \downarrow\}$$

so that

$$G_m(\vec{x}) = \{t^M[x_1, \ldots, x_n] \in M \mid t(v_1, \ldots, v_n) \text{ is a term of depth } \leq m\}$$

$$(\mathbf{M} \upharpoonright \bigcup_m G_m(\vec{x})) \text{ is the subalgebra generated by } \vec{x}$$
I The Locality Axiom

An algorithm $\alpha$ of arity $n$ of an algebra $M = (M, 0, 1, \Phi^M)$ assigns to each subalgebra $U \subseteq_p M$ an $n$-ary, strict partial function

$$\bar{\alpha}^U : U^n \rightarrow U$$

$\blacktriangleright$ $M$-algorithms “compute” strict partial functions, and they can be localized (relativized) to arbitrary subalgebras of $M$

We write

$$U \models \bar{\alpha}(\vec{x}) = w \iff \vec{x} \in U^n, w \in U \text{ and } \bar{\alpha}^U(\vec{x}) = w$$
II The Embedding Axiom

If $\alpha$ is an $n$-ary algorithm of $M$, $U, V \subseteq_p M$, and $\iota : U \hookrightarrow V$ is an embedding, then

$$U \models \overline{\alpha}(\bar{x}) = w \implies V \models \overline{\alpha}(\iota\bar{x}) = \iota w \quad (x_1, \ldots, x_n, w \in U)$$

In particular, if $U \subseteq_p M$, then $\overline{\alpha}^U \subseteq \overline{\alpha}^M$

- An algorithm treats the primitives of $M$ as oracles: it can request values $\phi^M(\bar{y})$, and use them if they are provided
III The Finiteness Axiom

If $\alpha$ is an $n$-ary algorithm of $M$, then

$$M \models \bar{\alpha}(\vec{x}) = w \implies \text{there is an } m \text{ such that } \vec{x}, w \in G_m(\vec{x})$$

and $M \upharpoonright G_m(\vec{x}) \models \bar{\alpha}(\vec{x}) = w$

In particular,

$$\bar{\alpha}^M(\vec{x}) \downarrow \implies \bar{\alpha}(\vec{x}) \in \bigcup_m G_m(\vec{x})$$

“‘The computation’ of $\bar{\alpha}^M(\vec{x})$ takes place within the subalgebra of $M$ generated by the input, and it is finite: take $m$ large enough so that every $y$ used in ‘the computation’ is in $G_m(\vec{x})$"
All algorithms—really—satisfy these axioms

- Explicit computation: $\alpha^U(\vec{x}) = t^U[\vec{x}]$, where $t(\vec{v})$ is a $\Phi$-term
- $\alpha^U$ is the partial function computed a fixed recursive (McCarthy) program $A$ in the signature $\Phi$ (as in Lecture 1)
- $\alpha^U$ is computed by a register machine (or RAM, or Turing machine or ...) from $\Phi^U$
- $\alpha^U$ is computed in Plotkin’s PCF above the algebra $U$
- $\alpha^U$ by computed in non-deterministic versions of any of these
Axioms for elementary algorithms

- **I, Locality Axiom:** An algorithm $\alpha$ of arity $n$ of an algebra $\mathcal{M} = (\mathcal{M}, 0, 1, \Phi^\mathcal{M})$ assigns to each subalgebra $U \subseteq_{p} \mathcal{M}$ an $n$-ary, strict partial function $\alpha^U : U^n \to U$ \hspace{1cm} ($U \models \alpha(\vec{x}) = w \iff \alpha^U(\vec{x}) = w$)

- **II, Embedding Axiom:** If $U, V \subseteq_{p} \mathcal{M}$, and $\iota : U \hookrightarrow V$ is an embedding, then

\[ U \models \alpha(\vec{x}) = w \implies V \models \alpha(\iota\vec{x}) = \iota w \quad (x_1, \ldots, x_n, w \in U) \]

- **III, Finiteness Axiom:**

\[ \mathcal{M} \models \alpha(\vec{x}) = w \implies there \ is \ an \ m \ such \ that \ \vec{x}, w \in G_m(\vec{x}) \]

and $\mathcal{M} \upharpoonright G_m(\vec{x}) \models \alpha(\vec{x}) = w$
The embedding complexity of an algorithm

If $\alpha$ is an algorithm of $M$ and $M \models \overline{\alpha}(\vec{x}) = w$, set

$$c^\ell_{\alpha}(\vec{x}) = \text{the least } m \text{ such that } M \upharpoonright G_m(\vec{x}) \models \overline{\alpha}(\vec{x}) = w$$

This is defined by the Finiteness Axiom

- Intuitively, if $m = c^\ell_{\alpha}(\vec{x})$, then any implementation of $\alpha$ will need to “consider” (use) some $u \in M$ of depth $m$; and so it will need at least $m$ steps to construct this $u$ from the input using the primitives
- If $\overline{\alpha}(\vec{x}) = t^M[\vec{x}]$, then $c^\ell_{\alpha}(\vec{x}) \leq \text{depth}(t(\vec{v}))$
- $c^\ell_{\alpha}$ is majorized by all usual time-complexity measures, including the number of calls to the primitives
The embedding complexity of a (computable) function

Fix \( f : M^n \to M \). An embedding \( \iota : M \upharpoonright G_m(\vec{x}) \hookrightarrow M \) respects \( f \) at \( \vec{x} \) if

\[
f(\vec{x}) \in G_m(\vec{x}) \quad \text{&} \quad \iota(f(\vec{x})) = f(\iota(\vec{x}))
\]

**Lemma**

*If some algorithm computes \( f \) in \( M \), then for each \( \vec{x} \), there is some \( m \) such that every embedding \( \iota : M \upharpoonright G_m(\vec{x}) \hookrightarrow M \) respects \( f \) at \( \vec{x} \)*

**Proof** Take \( m = c^\iota_\alpha(\vec{x}) \) for some \( \alpha \) such that \( f = \bar{\alpha}^M \)

\[
c^\iota_f(\vec{x}) = \text{the least } m \text{ such that every } \iota : M \upharpoonright G_m(\vec{x}) \hookrightarrow M \text{ respects } f \text{ at } \vec{x}
\]

If \( \alpha \) computes \( f \) in \( M \), then \( c^\iota_f(\vec{x}) \leq c^\iota_\alpha(\vec{x}) \)

- To show that \( m \) is an absolute lower bound for the computation of \( f(\vec{x}) \) show that \( f(\vec{x}) \not\in G_m(\vec{x}) \),

or construct \( \iota : M \upharpoonright G_m(\vec{x}) \hookrightarrow M \) such that \( \iota f(\vec{x}) \neq f(\iota \vec{x}) \)
Outline of a proof

**Theorem** (van den Dries, ynm)

*For the algebra* $\mathbf{M} = (\mathbb{N}, 0, 1, \leq, +, -, \cdot, \text{iq}, \text{rem})$ *and the relation of coprimeness* $x \perp y$,

\[ a^2 = 1 + 2b^2 \implies c_\perp (a, b) > \frac{1}{10} \log \log(a) \quad (*) \]

*So if* $\alpha$ *decides coprimeness in* $\mathbf{M}$, *then* $(*)$ *holds with* $c_\alpha (a, b)$

- If $2^{2^{4m+6}} \leq a$, then every $X \in G_m(a, b)$ can be written uniquely as

\[ X = \frac{x_0 + x_1 a + x_2 b}{x_3} \quad \text{with} \quad x_i \in \mathbb{Z}, \quad |x_i| \leq 2^{2^{4m}} \]

and we can define $\iota : \mathbf{M} \upharpoonright G_m(a, b) \hookrightarrow \mathbf{M}$ *using* $\lambda = 1 + a!$,

\[ \iota(X) = \frac{x_0 + x_1 \lambda a + x_2 \lambda b}{x_3}, \quad \text{so} \quad (\iota(a), \iota(b)) = (\lambda a, \lambda b) \]
\[ M = (\mathbb{N}, 0, 1, \text{Parity}, \text{iq}_2, \leq, +, \div, \text{Presburger functions}) \]

- (van den Dries, ynm) *If* \( R(x) \) *is one of the relations*

  \[ x \text{ is prime, } x \text{ is a perfect square, } x \text{ is square free,} \]

  *then for some* \( r > 0 \) \*and infinitely many* \( a, c \)

  \[ c^\ell_R(a) > r \log(a) \]

- (van den Dries, ynm) *For some* \( r > 0 \) \*and infinitely many* \( a, b \)

  \[ c^\ell_\parallel(a, b) > r \log(\max(a, b)) \]

- (Joe Busch) *If* \( R(x, p) \iff x \text{ is a square mod } p \),

  *then for some* \( r > 0 \) \*and a sequence* \( (a_n, p_n) \) \*with* \( p_n \to \infty \),

  \[ c^\ell_R(a_n, p_n) > r \log(p_n) \]

In the last two examples, the results match up to a multiplicative constant the known binary algorithms, so these are optimal.
Primality in $\mathbf{M} = (\mathbb{N}, 0, 1, \text{Parity}, \text{iq}_2, \leq, +, -, \text{Presburger})$

**Theorem** (van den Dries, ynm)

If $\text{Prime}(p) \iff p$ is prime, then in $\mathbf{M}$, for some $r > 0$ and all primes $p$,

$$c_{\text{Prime}}(p) > r \log p \tag{\ast}$$

So if $\alpha$ decides primality in $\mathbf{M}$, then $(\ast)$ holds with $c_{\alpha}(p)$

- If $2^{2m+2} \leq a$, then every $X \in G_m(a)$ can be written uniquely as

$$X = \frac{x_0 + x_1 a}{2^m} \quad \text{with } |x_i| \leq 2^m,$$

and we can define $\iota : \mathbf{M} \upharpoonright G_m(a) \rightarrow \mathbf{M}$ by

$$\iota(X) = \frac{x_0 + x_1 \lambda a}{2^m}, \quad \text{with } \lambda = 1 + 2^m, \text{ so } \iota(a) = \lambda a$$
Primality in binary

- If \( \text{Prime}(p) \iff p \) is prime, then in
  \[
  N_b = (\mathbb{N}, 0, 1, \text{Parity}, \text{iq}_2, (x \mapsto 2x), (x \mapsto 2x + 1))
  \]
  for some \( r > 0 \) and all primes \( p \),
  \[
  c^i_{\text{Prime}}(p) \geq r \log p \quad (*)
  \]

- This should follow trivially from number-theoretic results, because it takes at least \( i \) applications of the primitives of \( N_b \) to read \( i \) bits of the input; we should have \( r = 1 \)

- **Theorem** (Tao). *For infinitely many primes \( p \), if \( p' \) is constructed by changing any bit in the binary expansion of \( p \) except the highest, then \( p' \) is not prime*

- Tao found subsequently that this result is implicit in a paper of Cohen and Selfridge from 1975 and explicitly noted in a 2000 paper by Sun, and he obtained more general results
Non-uniform complexity

What if you are only interested in deciding $R(\vec{x})$ for $n$-bit numbers ($< 2^n$) and you are willing to use a different algorithm for each $n$?

**Theorem** (The lookup algorithm)

*For each $k$-ary relation $R$ on $\mathbb{N}$ and each $n$, there is an $\mathbb{N}_b$-term (with conditionals) $t_n(\vec{v})$ of depth $\leq n = \log_2(2^n)$ which decides $R(\vec{x})$ for all $\vec{x} < 2^n$.***

- Non-uniform lower bounds are never greater than log
- The best ones establish the optimality of the lookup algorithm (and are most interesting when some uniform algorithm matches the lookup up to a multiplicative constant)
- They are mostly about “the size” of $t(\vec{v})$
- They do not follow from Axiom I – III
Recursive (McCarthy) programs of \( \mathbf{M} = (M, 0, 1, \Phi^M) \)

Explicit \( \Phi \)-terms (with \( p^n_i \) partial function variables)

\[
A \equiv 0 \mid 1 \mid v_i \mid \phi(A_1, \ldots, A_n) \mid p^n_i(A_1, \ldots, A_n) \\
| \text{if } (A_0 = 0) \text{ then } A_1 \text{ else } A_2,
\]

Recursive program (only \( \vec{x}_i, p_1, \ldots, p_K \) occur in each part \( A_i \)):

\[
A : \begin{cases}
p_A(\vec{x}_0) = A_0 \\
p_1(\vec{x}_1) = A_1 \\
\vdots \\
p_K(\vec{x}_K) = A_K
\end{cases}
\]

\( A_0 : \text{the head}, (A_1, \ldots, A_K) : \text{the body} \)

The elementary algorithms of \( \mathbf{M} \) are expressed by recursive programs

(and they satisfy Axioms I – III)
A non-uniform lower bound result for elementary algorithms

If $\alpha$ is the algorithm expressed by a recursive program in $\mathbf{M}$, let

$$c^s_{\alpha}(\vec{x}) = \text{the number of calls to the primitives}$$

made in the computation of $\overline{\alpha}(\vec{x}) \geq c^l_{\alpha}(\vec{x})$

**Theorem** (van den Dries, ynm)

Let $\mathbf{M} = (\mathbb{N}, 0, 1, \leq, +, -, \cdot, \text{iq}, \text{rem})$. There is some $r > 0$, such that for all sufficiently large $n$ and every $\mathbf{M}$-elementary algorithm $\alpha$ which decides coprimeness for all $x, y < 2^n$, there exist $a, b < 2^n$ such that

$$c^s_{\alpha}(a, b) > r \log_2 n \geq r \log_2 \log_2(\max(a, b))$$

The proof is by the embedding method, but uses special properties of recursive programs (the **computation space**).
Logical extensions (a la Tarski)

A \((\Phi \cup \Psi)\)-algebra \(\overline{M}\) is a logical extension of a \(\Phi\)-algebra \(M\) if

1. \(M \subseteq \overline{M}\), \(0^M = 0^{\overline{M}}\), \(1^M = 1^{\overline{M}}\)

2. For each \(\phi \in \Phi\), \(\phi^M = \phi^{\overline{M}}\)

3. Every bijection \(\iota : M \rightarrow M\) which fixes \(0, 1\) can be extended to a bijection \(\overline{\iota} : \overline{M} \rightarrow \overline{M}\) such that for every \(\psi \in \Psi\),

\[\psi^{\overline{M}}(\overline{\iota x}) = \overline{\iota} \psi^{\overline{M}}(\overline{x})\quad (\overline{x} \in \overline{M}^n)\]

i.e., \(\iota\) is an automorphism of the reduct \((\overline{M}, 0, 1, \psi^{\overline{M}})\)

Random Access (and all other kinds of) Machines from \(\Phi^M\), Plotkin’s PCF over \(M\), etc., are all faithfully represented by recursive programs on logical extensions of \(M\)
The persistence of embedding complexity

**Theorem** (van den Dries, Neeman, ynm)

If \( f : M^n \rightarrow M \) and \( \overline{M} \) is a logical extension of \( M \), then

\[
c_f^l(\vec{x}, M) = c_f^l(\vec{x}, \overline{M}) \quad (\vec{x} \in M^n)
\]

- This is why the embedding method gives the same lower bounds (for a function \( f \) from specified primitives) for RAMs and for recursive programs, even though the direct simulation of RAMs by recursive programs has an overhead.
- The basic non-uniform results obtained by the embedding method also extend to arbitrary logical extensions.
Theorem

If \( \leq \) is an ordering of a set \( A \), \( \overline{A} \) is a logical extension of \((A \cup \{0, 1\}, 0, 1, \leq)\) such that \( A^* \subseteq A \), and \( \alpha \) is an elementary algorithm of \( \overline{A} \) which sorts the sequences in \( A^* \), then

\[
|u| = n \implies c^s_\alpha(u) \geq \log_2(n!) \sim n \log_2(n),
\]

where \( c^s_\alpha(u) \) is the number of comparisons made by \( \alpha \) in the computation of \( \text{sort}(u) \)

This is proved by the classical, counting argument