Prelim in Philosophy of Mathematics

Philosophy of mathematics and mathematical practice in the foundations of the calculus.

The topic of examination focuses on the interaction between philosophy and mathematical practice during the period comprising the emergence of the calculus (mid-seventeenth century) to its final rigorization in terms of epsilon-delta methods in the late nineteenth century. In addition, we also pursue a number of topics that are discussed in the contemporary literature and that can be traced directly back to these earlier developments, e.g. structuralism and purity of methods.

One important aspect in the development of the calculus is the transition from a purely geometrical approach to an analytic/algebraic presentation. The change that led from a purely geometrical treatment of the continuum to an arithmetized one (achieved by Dedekind) was technically difficult and philosophically significant.

Our readings begin with a theorem by Torricelli to the effect that a certain infinitely long solid can be shown to have finite volume. While this is a standard ordinary result of the integral calculus, Torricelli's treatment is completely geometrical. He uses both Archimedean and Cavalierian (indivisibilist) techniques to prove the result in question. The latter techniques and the result itself require appeal to the infinite as 'actual' infinite (as opposed to 'potential' infinite). This result therefore challenged empiricist philosophies of mathematics (such as those of Hobbes and Gassendi) who had to struggle, unsuccessfully, to account philosophically for this new mathematical development.

We continue with a reading of the Analyst by Bishop Berkeley. By the time Berkeley wrote the Analyst, the calculus had developed into a powerful mathematical theory but its foundations were still problematic, for they appealed to notions such as 'infinitesimal quantity' that were not rigorously grounded. Berkeley's criticism of the calculus rests on several (logical, epistemological, and ontological) arguments. In addition, Berkeley offered his theory of double mistakes to account for why the analysts, despite the shaky foundation of their discipline, arrived systematically at truths and rarely made mistakes.

No account of the philosophy of mathematics in the eighteenth century and after can bypass the fundamental importance of Kant's philosophy of mathematics. Using selections from the first Critique and the Prolegomena, we investigated the two fundamental theses of Kant's philosophy of mathematics: 1) mathematics is synthetic a priori; 2) mathematics, unlike philosophy, rests on construction of concepts in intuition.

Kant's position is attacked by Bolzano, one of the great mathematicians and philosophers of the nineteenth century. Bolzano wrote a fundamental paper in 1817 in which he proved the intermediate value theorem with the epsilon-delta style we are nowadays accustomed to. (Cauchy was also a key player in this reshaping of the calculus). But Bolzano’s memoir is more than great piece of mathematics. Bolzano uses his paper on the intermediate value theorem as a sample of an anti-Kantian philosophical program meant to show that mathematics, just like philosophy, can be developed from analysis of concepts without the postulation of a priori intuition of space and time. Bolzano’s memoirs is also a great example of a purity program in mathematical practice, namely the attempt to remove any appeal to kinematics and geometry from fundamental theorems of analysis.

As we saw, Bolzano also helped rigorize arguments in analysis involving limits, pointing the way to modern epsilon-delta formulations. Another step in this direction was taken by Dedekind, who bridged a gap in rigor by explicitly stating the assumptions regarding the number system involved. Indeed, he provided a construction of the real numbers, starting from the rationals (and ultimately the naturals). This raised other questions, regarding the nature of these numbers systems, given their seemingly arbitrary constructions and multitude of models. This led Dedekind to a certain kind of structuralist view of mathematical objects (see also Reck’s 2003 paper). The various threads of philosophical concerns raised by the historical figures above have been woven into modern philosophical considerations. Directly continuing Dedekind’s line of thought are modern day structuralists, who seek to provide a mathematical ontology that matches up with mathematical methodology (see Reck and Price 2000, Reck 2003, Benacerraf 1965). Another line of investigation continues the concerns, especially of Bolzano and Dedekind, regarding the separation of subfields of mathematics--this is the study of purity of methods (see Detlefsen 2008).

We conclude our readings with Benacerraf 1970 as an example of an attempt to make salient the desiderata, often in conflict, for a unified philosophy of mathematics which would bring together ontology, epistemology and semantics in a unified framework.

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