1. Prove or disprove: For any uncountable well-ordered set \((X, <)\) there is a countable well-ordered set \((Y, <)\) for which \((X, <) \equiv (Y, <)\). 

2. Suppose that \(L\) is a first-order language having only finitely many nonlogical symbols and that \(T\) is a theory in \(L\) having no uncountable models. Show that up to isomorphism \(T\) has only finitely many models. 

3. Show that there is some \(e \in \omega\) for which \((\forall x \in \omega) x \in W_e \iff (x + e + 1) \in W_e\). Show, moreover, that any such \(W_e\) is recursive. 

4. Let \(\mathfrak{N}\) be a nonstandard model of Peano arithmetic. Show that there is an element \(a \in \mathfrak{N}\) such that for any standard prime number \(p\), \(p^p\) divides \(a\) and \(a/p^p\) is coprime to \(p\). 

5. Show that there is a total recursive function \(f : \omega \rightarrow \omega\) such that for all \(e \in \omega\) the set \(W_e\) is finite if and only if \(\omega \setminus W_{f(e)}\) is finite. 

6. Consider the structure \((\omega, S)\) where \(S : \omega \rightarrow \omega\) is the successor function \(x \mapsto x + 1\). Let \(T := \text{Th}(\omega, S)\) be the complete theory of this structure. How many 3-types (over \(\emptyset\)) are there relative to \(T\)? Describe all of the 3-types giving isolating formulas where possible. 

7. Let \(\text{Pr}_{\text{PA}}(x)\) be the usual formula which naturally expresses that the sentence encoded by \(x\) is provable from Peano arithmetic. Let \(\phi(x)\) be a formula in the language of arithmetic in the one free variable \(x\). Let \(\text{Sub}_\phi\) be the definable (relative to Peano arithmetic) function which takes a number \(a\) and returns the code for the sentence obtained by substituting \(a\) for \(x\) in \(\phi\). Show that if \(\text{PA} \vdash (\forall z)(\text{Pr}_{\text{PA}}(\text{Sub}_\phi(z)) \rightarrow \phi(z))\), then \(\text{PA} \vdash (\forall z)\phi(z)\). 

8. Let \(L\) be a first order language and \(\mathfrak{A}\) and \(L\)-structure with universe \(A\). Let \(L'\) be obtained from \(L\) by adjoining one new one place relation symbol \(S\). We say that \(S \subseteq A\) is implicitly definable if there is an \(L'\) sentence \(\sigma\) for which \((\mathfrak{A}, S') \models \sigma \iff S = S'\). 

Is it the case that whenever a set \(S\) is implicitly definable in some \(L\)-structure, then it must be explicitly (ie in \(L\)) definable? Prove that your answer is correct. 

9. Show that there is a nonstandard model \(\mathfrak{N}\) of Peano arithmetic having no proper elementary submodels.