GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE
PRELIMINARY EXAMINATION

There are eight questions. Partial credit may be assigned for substantially correct partially worked solutions. To pass, you need a score of roughly fifty percent, though the ultimate decision about the passing mark will be decided by the committee of graders.

1. Prove or refute:
   (a) If $A$ and $B$ are disjoint $\Sigma^0_1$ subsets of $\omega$, then there is a $\Delta^0_1$ set $C$ such that $A \subseteq C$ and $C$ is disjoint from $B$.
   (b) If $A$ and $B$ are disjoint $\Pi^0_1$ subsets of $\omega$, then there is a $\Delta^0_1$ set $C$ such that $A \subseteq C$ and $C$ is disjoint from $B$.

2. Let $T$ be a decidable theory in a finite language, and suppose all models of $T$ are infinite. Show $T$ has a model $\mathfrak{A}$ with universe $\omega$ such that the satisfaction relation $\{ (\phi, \overline{m}) | \mathfrak{A} \models \phi(\overline{m}) \}$ is recursive.

3. Show that there is a model $\mathfrak{M} \models \text{PA}$ of Peano arithmetic and $a \in |\mathfrak{M}| \setminus \mathbb{N}$ a nonstandard element of the universe of $\mathfrak{M}$ which is definable.

4. Let $\mathfrak{A}$ be an $L$-structure and two elements $a$ and $b$ of the universe of $\mathfrak{A}$. Show that the following are equivalent.
   (a) There is a definable function $f$ for which $f(a) = b$.
   (b) For any elementary extension $\mathfrak{B} \succeq \mathfrak{A}$ and automorphism $\sigma : \mathfrak{B} \to \mathfrak{B}$, if $\sigma(a) = a$, then $\sigma(b) = b$.

5. Let $L = L(U, V)$ be the first-order language having exactly two unary predicate symbols, $U$ and $V$, and no other nonlogical symbols. Describe all the of the complete theories in $L$. You should show that the theories you propose are distinct and that they exhaust all of the completions.

6. Let $\{ W_e \}_{e \in \omega}$ be the usual enumeration of the recursively enumerable sets. Show that $\text{Fin} := \{ e \in \omega : W_e \text{ is finite} \}$ is $\Sigma^0_2$-complete.

7. Let $\mathfrak{A} = (U, I, f, g, \ldots)$ be a structure for a finite language $L$, where $I$ is a unary relation and $f$ and $g$ are binary functions. Let $\pi$ be an isomorphism from $(\mathbb{N}, +, \times)$ to $(I, f \upharpoonright I, g \upharpoonright I)$. For $\phi$ an $L$-sentence, let $\text{GN}(\phi)$ be the Gödel number of $\phi$ in some reasonable Gödel numbering.
   Show that $\{ \pi(\text{GN}(\phi)) | \phi \text{ is an } L\text{-sentence and } \mathfrak{A} \models \phi \}$ is not definable over $\mathfrak{A}$ without parameters.

8. Give an example of a pair of first order languages $L \subseteq L'$ and complete theories $T \subseteq T'$ in $L$ and $L'$, respectively, for which $T'$ is $\aleph_1$-categorical but $T$ has more than one model of cardinality $\aleph_1$. [Prove that your proposed example works.]

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