LOGIC AND THE METHODOLOGY OF SCIENCE
PRELIMINARY EXAMINATION

1. Let $\mathcal{L} = \mathcal{L}(F_1)$ be the first-order language having exactly one unary function symbol and no constant or relation symbols. Give examples of infinite $\mathcal{L}$-structures $\mathfrak{M} = (M, F_1^{\mathfrak{M}})$ for each of the following properties (and prove that each of your proposed examples actually has the stated property).

(a) $\mathfrak{M}$ has no nontrivial automorphism.
(b) $\mathfrak{M}$ has a countably infinite set of automorphisms.
(c) $\mathfrak{M}$ has uncountably many automorphisms, but for each $a \in M$ there are only finitely many $b \in M$ for which there is an automorphism σ of $\mathfrak{M}$ with $\sigma(a) = b$.

2. Let $\mathcal{L} = \mathcal{L}(R)$ be the first-order language with exactly one binary relation symbol and no constant or function symbols.

(a) Let T be a consistent arithmetic $\mathcal{L}$-theory. Show that there is an arithmetic set $U \subseteq \omega$ and an arithmetic $S \subseteq \omega \times \omega$ for which $(U, S) \models T$.
(b) Let $S \subseteq \omega \times \omega$ be arithmetic. Show that $\mathrm{Th}(\omega, S)$ is recursive in $\mathbf{0}^{(\omega)}$.

3. Let $\mathcal{L}$ be a first-order language and T a complete and consistent $\mathcal{L}$-theory.

We say that T has a Vaughtian pair if there is an elementary extension $\mathfrak{M} \prec \mathfrak{N} \models T$ of models of T and some formula $\varphi(x_1, \ldots, x_n) \in \mathcal{L}_M$ in the free variables $(x_1, \ldots, x_n)$ possibly with parameters from M for which $\{\boldsymbol{a} \in M^n : \mathfrak{M} \models \varphi(\boldsymbol{a})\} =: \varphi(\mathfrak{M})$ is infinite but $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$ while $\mathfrak{M} \neq \mathfrak{N}$.

Show that if T has no Vaughtian pairs, then for every formula $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ there is a natural number $N = N_\varphi$ so that for any model $\mathfrak{M} \models T$ and any choice of parameters $\boldsymbol{b} \in M^m$, if $\varphi(\mathfrak{M}; \boldsymbol{b})$ is finite, then $|\varphi(\mathfrak{M}; \boldsymbol{b})| \leq N$.

4.

(a) Show that there is a total Δ^0_2 function $f : \omega \to \omega$ for which for every recursive function $g : \omega \to \omega$ there is a number $m \in \omega$ such that for all $n > m$ we have $f(n) > g(n)$.
(b) Show that any such function has Turing degree greater than or equal to $\mathbf{0}'$.

5. Suppose that $\mathfrak{M}$ is a countably infinite model of a countable $\aleph_0$-categorical theory. Must $\mathfrak{M}$ have a nontrivial automorphism? (Prove your answer.) [Bonus: What happens for uncountable theories? Again, prove your answer.]

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6. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be an extension of first-order languages. Let $\mathfrak{M}$ be an infinite $\mathcal{L}_2$-structure and let $\mathfrak{N} := (\mathfrak{M} \upharpoonright \mathcal{L}_1)$ be the reduct of $\mathfrak{M}$ to $\mathcal{L}_1$. Let $T_2 := \mathrm{Th}_{\mathcal{L}_2}(\mathfrak{M})$ be the theory of $\mathfrak{M}$ in $\mathcal{L}_2$ and $T_1 := \mathrm{Th}_{\mathcal{L}_1}(\mathfrak{N})$ be the theory of $\mathfrak{N}$ in $\mathcal{L}_1$.

(a) Prove or disprove: If T_2 is $\aleph_0$-categorical, then T_1 is $\aleph_0$-categorical.
(b) Prove or disprove: If T_2 is $\aleph_1$-categorical, then T_1 is $\aleph_1$-categorical.

7. In what follows, the word "formula" means "formula in $\mathcal{L}(+, \times, 0, 1, <)$, the language of ordered rings." In particular, a formula has no constant symbols other than 0 and 1.

(a) Let $\mathfrak{N} \models \mathrm{PA}$ be a nonstandard model of Peano Arithmetic. Show that there is an element $a \in N$ such that for every quantifier free formula $\phi(x)$ if $\mathfrak{N} \models (\exists x)\phi(x)$, then $\mathfrak{N} \models (\exists x)[x < a \;\&\; \phi(x)]$.
(b) Show that if T is a consistent extension of PA for which $(\mathbb{N}, +, \times) \not\models T$, then there exists a model $\mathfrak{N} \models T$ for which there is **no** $a \in N$ such that for all formulas $\phi(x)$ we have $\mathfrak{N} \models (\exists x)\phi(x) \iff \mathfrak{N} \models (\exists x)[x < a \;\&\; \phi(x)]$.

8. Let $\mathcal{L} = \mathcal{L}(F)$ be the first-order language with a single unary function symbol F and no constant or relation symbols. Let $\mathfrak{M}$ be the $\mathcal{L}$-structure having universe $\{0, 1, 2, 3, 4\}$ and $F^{\mathfrak{M}}(0) := 1$, $F^{\mathfrak{M}}(1) := 2$, $F^{\mathfrak{M}}(2) := 0$, $F^{\mathfrak{M}}(3) = 4$ and $F^{\mathfrak{M}}(4) = 3$.

(a) Which subsets of M are $\mathcal{L}$-definable without parameters?
(b) Which subsets of M are $\mathcal{L}$-definable with the parameter 0?

9. W_e is the e^{th} recursively enumerable set under the standard enumeration. Let $A := \{e \in \omega : (\omega \smallsetminus W_e)$ contains an infinite r.e. set$\}$ and $B := \{e \in \omega : (\omega \smallsetminus W_e)$ is finite $\}$.

(a) Let C be a Π^0_2 set. **Show** that there is a total recursive function f such that for all c, if $c \in C$, then $f(c) \in A$ and if $c \notin C$, then $f(c) \in B$.
(b) Show that A is Σ^0_3 complete.