Logic and the Methodology of Science
June 2003 Preliminary Exam

August 23, 2005

1. Let $\mathcal{L}$ be a finite first-order language whose formulae have been Godel numbered in some natural way. Let Sat be the set of all (Godel numbers of) satisfiable $\mathcal{L}$-formulae.

(a) Show that Sat is Π^0_1.

(b) Give an example of a finite $\mathcal{L}$ for which Sat is not recursive. Outline a proof that your example works.

2. Recall that a sentence is *universal* if it is of the form $(\forall x_1)\cdots(\forall x_n)\theta(\vec{x})$ where θ is quantifier-free. We say that T is *universal* if there is a set U of universal $\mathcal{L}$-sentences for which $T \vdash\dashv U$. Show that T is universal if and only if for any model $\mathfrak{M} \models T$ and any substructure $\mathfrak{N} \subseteq \mathfrak{M}$ one has $\mathfrak{N} \models T$.

3. Let $\langle W_e \mid e < \omega\rangle$ be a standard enumeration of the recursively enumerable sets. Show that $\{e \mid W_e \text{ is finite }\}$ is Σ^0_2-complete.

4. Let $\mathcal{L}$ be a first-order language and $\mathfrak{M}$ an $\mathcal{L}$-structure. Recall that a sequence $\langle a_i \mid i \in \omega\rangle$ of elements of $M = |\mathfrak{M}|$ is *indiscernible* for $\mathfrak{M}$ if for any natural number n, any $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$, and pair of increasing n-tuples $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ of natural numbers, we have $\mathfrak{M} \models \psi[a_{i_1}, \ldots, a_{i_n}]$ iff $\mathfrak{M} \models \psi[a_{j_1}, \ldots, a_{j_n}]$. Show that if $\mathfrak{M}$ is infinite, there is some elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and an indiscernible sequence $\langle a_i \mid i \in \omega\rangle$ for $\mathfrak{N}$ with $a_0 \neq a_1$.

Show by example that the elementary extension may be necessary.

5. For $A \subseteq \omega\times\omega$ let $A_a = \{b \mid \langle a, b\rangle \in A\}$.

(a) Let A be recursively enumerable (r.e.), and suppose $n < \omega$ is such that A_a has size n for all a. Show that A is recursive.

(b) For each $n > m$, give an example of an r.e. set A such that for all a, A_a has size n or size m, but A is not recursive.

6. Let $\mathcal{L}$ be a first-order language and $\mathfrak{M}$ an $\mathcal{L}$-structure.

a Suppose that there are only finitely many orbits in $M = |\mathfrak{M}|$ under the automorphism group of $\mathfrak{M}$. (Such an orbit is an equivalence class of the equivalence relation: xEy iff there is an automorphism π of $\mathfrak{M}$ such that $\pi(x) = y$.) Show that there are finitely many formulas $\psi_1(x), \ldots, \psi_n(x)$ in one free variable x such that for any $\mathcal{L}$-formula $\vartheta(x)$ in one free variable there is some $i \leq n$ with $\mathfrak{M} \models (\forall x)\ \vartheta(x) \leftrightarrow \psi_i(x)$.

b Is the converse true? Prove or provide (with proof) a counter-example.

c Assume that $\mathcal{L}$ and $\mathfrak{M}$ are both countable and for each natural number m there is a finite sequence $\phi_1^m(x_1, \ldots, x_m), \ldots, \phi_{n_m}^m(x_1, \ldots, x_m)$ of formulas in m free variables such that for any other formula $\theta(x_1, \ldots, x_m)$ there is some $i \leq n_m$ with $\mathfrak{M} \models (\forall x_1, \ldots, x_m)\ \theta(\vec{x}) \leftrightarrow \phi_i^m(\vec{x})$.

Show that there are only finitely many orbits in M under the action of the automorphism group of $\mathfrak{M}$.

7. Let $\mathcal{N} = (\omega, +, \cdot, S, <, 0)$ be the standard structure of arithmetic. Let $\mathcal{N} \prec \mathcal{M}$, and $\mathcal{N} \neq \mathcal{M}$. Let M be the universe of $\mathcal{M}$. Suppose

$$(\forall x \in M)(\exists y \in \omega)(\forall z \in M)(\exists t \in \omega)\mathcal{M} \models \phi[x, y, z, t].$$

Show that for some $m < \omega$,

$$(\forall x \in M((\exists y < m)(\forall z \in M)(\exists t < m)\mathcal{M} \models \phi[x, y, z, t].$$

8. Let $\Phi = \{\phi_e \mid e \in \omega\}$ be the set of all partial recursive functions of one variable, equipped with some standard enumeration. Let $F \colon \Psi \to \omega$, where $\Psi \subseteq \Phi$, and let f be a partial recursive function such that

$$\mathrm{dom}(f) = \{e \mid \phi_e \in \Psi\},$$

and for all $e \in \mathrm{dom}(f)$,

$$f(e) = F(\phi_e).$$

(a) Show that if $\Psi = \Phi$ (so that f is total), then F is a constant function.

(b) (Harder.) Show that in any case, there is an r.e. collection $\mathcal{H}$ of finite partial functions such that

$$\Psi = \{\phi \in \Phi \mid \exists h \in \mathcal{H}(h \subseteq \phi)\}.$$

9. Recall that a formula $\phi(x, y)$ in the language of PA *represents* a relation $R \subseteq \omega \times \omega$ iff for all $n, m \in \omega$,

$$R(n, m) \Rightarrow \mathsf{PA} \vdash \phi(\bar{n}, \bar{m})$$

and

$$\neg R(n,m) \Rightarrow \mathsf{PA} \vdash \neg\phi(\bar{n},\bar{m}),$$

where $\bar{k}$ is the numeral for k. Let $\mathrm{Prov}(x,y)$ be a standard formula in the language of Peano Arithmetic (PA) representing the relation y is (the Godel number of) a proof of x from the axioms of PA. Similarly, let $\mathrm{neg}(x,y)$ be a standard formula representing: x and y are sentences, and one is the negation of the other. Let $\mathrm{Prov}^*(x,y)$ be the formula

$$\mathrm{Prov}(x,y) \wedge \forall z < y \forall w(\mathrm{neg}(x,w) \rightarrow \neg\mathrm{Prov}(w,z)).$$

(a) Show that $\mathrm{Prov}^*(x,y)$ represents over PA the same relation as does $\mathrm{Prov}(x,y)$.

(b) Let Con^* be the sentence

$$\forall x \forall w \forall y \forall z[((\mathrm{Prov}^*(x,y) \wedge \mathrm{Prov}^*(w,z)) \rightarrow \neg\mathrm{neg}(x,w)].$$

Show that PA proves Con^*.

(c) Explain where Godel's proof of the second incompleteness theorem breaks down, when applied to Con^*.