PRELIMINARY EXAMINATION
GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE

For the following questions we fix a standard enumeration $\langle \varphi_e : e \in \omega \rangle$ of the partial recursive functions with the corresponding enumeration $\langle W_e := \{x \in \omega : \varphi_e(x) \downarrow\} : e \in \omega \rangle$ of the recursively enumerable sets.

1. Suppose that $E \subseteq \omega \times \omega$ is a recursively enumerable equivalence relation having finitely many equivalence classes. Prove that E is recursive.

2. Prove that there is a recursively enumerable set having the property that its complement is infinite but it meets every infinite recursively enumerable set nontrivially.

3. Show that for any infinite model $\mathfrak{A}$ there is a proper elementary extension $\mathfrak{B} \succ \mathfrak{A}$ and an elementary embedding $f : \mathfrak{B} \to \mathfrak{B}$ for which $A = \bigcap_{n=1}^{\infty} f^n(B)$. [Hint: Ehrenfeucht-Mostowski]

4. Let $\mathcal{L}$ be the first-order language having one binary function symbol $+$ and a constant symbol 0. Prove that the extension of structures $(\mathbb{Q}, +, 0) \subseteq (\mathbb{R}, +, 0)$ is an elementary extension.

5. Show that there is no total function $f : \omega \to \omega$ for which $f \leq_T \varnothing'$ and for every $e \in \omega$ if W_e is finite, then $f(e) = \# W_e$.

6. Let $\mathcal{L}$ be a countable first-order language and $\mathcal{L}' := \mathcal{L}(\{P_i : i \in \omega\})$ the expansion of $\mathcal{L}$ by countably many new unary predicates. Let T' be a complete, consistent $\mathcal{L}'$-theory. Suppose that $\Sigma(x)$ is a set of $\mathcal{L}$ formulae in the free variable x and that for each $n \in \omega$ the restriction of T' to $\mathcal{L}(\{P_i : i < n\})$ has a model omitting $\Sigma(x)$. Prove that T' itself has a model omitting $\Sigma(x)$.

7. Consider the semiring

$$\mathbb{Z}[x]_{\geq 0} := \{f(x) \in \mathbb{Z}[x] : (\exists N \in \mathbb{N})(\forall n > N) f(n) \geq 0\}$$

Order $\mathbb{Z}[x]_{\geq 0}$ by

$$f \leq g \iff (\exists h \in \mathbb{Z}[x]_{\geq 0}) f + h = g$$

Prove or disprove: $(\mathbb{Z}[x]_{\geq 0}, \leq, +, \times, 0, 1) \models \mathrm{PA}$

8. Show by example (and prove that your example has the requisite properties) that there is a complete theory T in a countable language for which there is exactly one 1-type relative to T but continuum many 2-types.

Date: 22 January 2011.