Throughout this exam, we use the notation \{W_e\}_{e \in \mathbb{N}} for a standard listing of the recursively enumerable sets.

1. Prove or disprove: If \( \mathcal{M} \models \text{PA} \) is a model of Peano arithmetic and \( a \in |\mathcal{M}| \smallsetminus \mathbb{N} \) is a nonstandard element of the universe of \( \mathcal{M} \), then \( a \) is not definable.

2. Show that \( \{e: W_e \text{ is recursive}\} \) is a \( \Sigma^0_3 \) complete subset of \( \mathbb{N} \).

3. Let \( \mathcal{L} \) be a countable first-order language. Suppose that \( T \) is a consistent \( \mathcal{L} \)-theory having infinite models. Show that there is a model \( \mathcal{M} \models T \) of cardinality \( \aleph_1 \) in which at most \( \aleph_1 \) 1-types are realized.

4. Say that two sets \( U \) and \( V \) are recursively inseparable if there is no recursive set \( R \) such that \( U \subseteq R \) and \( V \subseteq \mathbb{N} \smallsetminus R \). (Here \( \mathbb{N} \smallsetminus R \) denotes the complement of \( R \) in \( \mathbb{N} \).)

   (1) Show that there is a pair of disjoint, recursively inseparable, recursively enumerable sets.

   (2) Show that any pair of disjoint \( \Pi^0_1 \) sets (ie. complements of recursively enumerable sets) is not recursively inseparable.

5. Let \( \mathcal{L} = \mathcal{L}(U) \) be the first-order language having one non-logical unary relation symbol, \( U \). Prove or disprove: the set of validities in \( \mathcal{L} \) is recursive (relative to the natural recursive encoding of \( \mathcal{L} \)).

6. Show that the class of existentially closed groups is not first-order axiomatizable.
   [Hint: If \( G \) is a group and \( \alpha : G \to G \) is an automorphism, then the semidirect product \( H := G \rtimes \mathbb{Z} \) in which \( (g, n) \cdot (h, m) = (g\alpha^n(h), n+m) \) is an extension group of \( G \).]

7. Suppose that \( f \) is a total recursive function. Prove or give a counter-example to each of the following.

   (1) There is an \( e \) such that \( W_{f(e)} = \{e\} \).

   (2) There is an \( e \) such that \( W_e = \{f(e)\} \).

8. Let \( T \) be an \( \aleph_0 \)-categorical theory in a countable language and \( \mathcal{M} \models T \) a countably infinite model of \( T \). What is the cardinality of the automorphism group of \( \mathcal{M} \)? (Prove that your answer is correct.)

9. Prove or disprove: The usual operation of addition is definable in the structure \((\mathbb{Q}, \cdot)\).