LOGIC AND THE METHODOLOGY OF SCIENCE
PRELIMINARY EXAMINATION

Throughout this exam, we use the notation $\{W_e\}_{e\in\mathbb{N}}$ for a standard listing of the recursively enumerable sets.

1. Prove or disprove: If $\mathfrak{N} \models \mathrm{PA}$ is a model of Peano arithmetic and $a \in |\mathfrak{N}| \smallsetminus \mathbb{N}$ is a nonstandard element of the universe of $\mathfrak{N}$, then a is *not* definable.

2. Show that {e: W_e is recursive} is a Σ^0_3 complete subset of $\mathbb{N}$.

3. Let $\mathcal{L}$ be a countable first-order language. Suppose that T is a consistent $\mathcal{L}$-theory having infinite models. Show that there is a model $\mathfrak{M} \models T$ of cardinality $\aleph_1$ in which at most $\aleph_0$ 1-types are realized.

4. Say that two sets U and V are recursively inseparable if there is no recursive set R such that $U \subseteq R$ and $V \subseteq \mathbb{N} \smallsetminus R$. (Here $\mathbb{N} \smallsetminus R$ denotes the complement of R in $\mathbb{N}$.)

(1) Show that there is a pair of disjoint, recursively inseparable, recursively enumerable sets.
(2) Show that any pair of disjoint Π^0_1 sets (ie. complements of recursively enumerable sets) is not recursively inseparable.

5. Let $\mathcal{L} = \mathcal{L}(U)$ be the first-order language having one non-logical unary relation symbol, U. Prove or disprove: the set of validities in $\mathcal{L}$ is recursive (relative to the natural recursive encoding of $\mathcal{L}$).

6. Show that the class of existentially closed groups is *not* first-order axiomatizable. [Hint: If G is a group and $\alpha : G \to G$ is an automorphism, then the semidirect product $H := G \rtimes \mathbb{Z}$ in which $(g, n)\cdot(h, m) = (g\alpha^n(h), n+m)$ is an extension group of G.]

7. Suppose that f is a total recursive function. Prove or give a counter-example to each of the following.

(1) There is an e such that $W_{f(e)} = \{e\}$.
(2) There is an e such that $W_e = \{f(e)\}$.

8. Let T be an $\aleph_0$-categorical theory in a countable language and $\mathfrak{M} \models T$ a countably infinite model of T. What is the cardinality of the automorphism group of $\mathfrak{M}$? (Prove that your answer is correct.)

9. Prove or disprove: The usual operation of addition is definable in the structure $(\mathbb{Q}, \cdot)$.

Date: 16 January 2008.