1. Let $\varphi(v)$ be a formula in the language of Peano Arithmetic (PA).

(a) Suppose that $\varphi(v)$ is $\Sigma_1$, and $\text{PA} \vdash \exists v \varphi(v)$. Show that $\text{PA} \vdash \varphi(\bar{n})$ for some numeral $\bar{n}$.

(b) Give an example of a formula $\varphi(v)$ such that $\text{PA} \vdash \exists v \varphi(v)$, but for all $n$, $\text{PA}$ does not prove $\varphi(\bar{n})$.

(c) Suppose $\varphi(v)$ is $\Sigma_1$ and $T$ is a consistent extension of $\text{PA}$ such that $T \vdash \exists v \varphi(v)$. Does it follow that $T \vdash \varphi(\bar{n})$ for some $n$.

2. Show there is a one-one 2-ary partial recursive function $\Psi$ such that for every one-one 1-ary partial recursive $f$, there is an $e$ such that for all $i$, $f(i) = \Psi(e, i)$.

3. Let $L$ be a finite language, and let $T$ be an axiomatizable $L$-theory. Fix a recursive enumeration of $T$, and let $T_n$ be the first $n$ sentences of $T$ in this enumeration. Suppose $M \models \text{PA}$ is such that $M \models \text{Con}(T_n)$ for all $n$. (On the right hand side, “$T_n$” should be interpreted as the numeral for the Godel number of $T_n$.) Show that $M$ interprets a model of $T$; that is, there is a model of $T$ whose universe, functions, and relations are all definable from parameters over $M$.

4. Let $E$ be an r.e. equivalence relation on $\omega$, and suppose $E$ is not recursive. Show
(a) \( E \) has infinitely many equivalence classes,

(b) for each \( n \), there are infinitely many equivalence classes whose cardinality is different from \( n \).

5. Let \( A \) be an r.e. set, and \( B = \{ e | W_e = A \} \). Show that either \( B \) is a \( \Delta^0_2 \) set, or \( B \) is a complete \( \Pi^0_2 \) set.

6. (a) A graph is a set with an irreflexive, symmetric binary relation. Show there is a graph \( G = (V, E) \) such that whenever \( J \) and \( K \) are disjoint finite subsets of \( V \), then there is an \( a \in G \) such that

\[
\forall b \in J(aEb) \text{ and } \forall b \in K(\neg aEb).
\]

(b) Show that if \( G \) is a graph as in part (a), then the theory of \( G \) is decidable.

7. Show that the theory of \((\mathbb{Q}, +)\) is decidable.

8. Let \( T \) be the theory of \((\mathbb{Z}, +)\). How many countable models (up to isomorphism) does \( T \) have?

9. Let \( T \) be a complete theory in a countable language. Show that the following are equivalent:

(a) \( T \) has a prime model \( A \) such that there is a \( B \prec A \) with \( B \neq A \),

(b) \( T \) has an uncountable atomic model.