Preliminary Examination
Group in Logic and the Methodology of Science
August 21, 2015

1. Let $\mathcal{L}_<$ be the first order language with with one binary relation symbol $<$. Let T be the set of sentences in $\mathcal{L}_<$ which are provable in first order Peano Arithmetic. Show that T is complete.

2. Prove that if all the elements of a first order structure $\mathfrak{M}$ are algebraic (*i.e.*, satisfy a formula with only finitely many solutions in $\mathfrak{M}$), then $\mathfrak{M}$ is atomic.

3. Prove that $A \leq_T 0'$ is low_2 (*i.e.* $A'' \equiv_T 0''$) if and only if there is a $0'$-computable function that dominates all A-computable functions.

4. Suppose that B is a Π^0_1 subset of ω and that B has no infinite recursive subset. Show that B is not many-one complete among Π^0_1 subsets of ω.

5. Show that there is a sentence φ in the language of Peano Arithmetic such that the following conditions hold.

   - $PA \vdash \varphi$.
   - The shortest proof from PA of φ uses at least 2^p symbols, where p is the number of symbols in φ.

6. Show that there are two dense linear orders without least or greatest elements $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same cardinality but $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not isomorphic.

7. Let $\mathcal{L}$ be a first-order language, T a complete $\mathcal{L}$-theory and Δ a set of $\mathcal{L}$-formulae in the free variable x. Show that the following two conditions are equivalent.

   a. For any model $\mathfrak{M} \models T$ and pair of elements a and b from the universe of $\mathfrak{M}$, if for all $\delta \in \Delta$ one has $\mathfrak{M} \models \delta(a) \leftrightarrow \delta(b)$, then for every $\mathcal{L}$-formula ψ in the free variable x one has $\mathfrak{M} \models \psi(a) \leftrightarrow \psi(b)$.

   b. For every $\mathcal{L}$ formula ψ in the free variable x there is a formula θ which is a finite Boolean combination of elements of Δ for which $T \vdash (\forall x)[\psi \leftrightarrow \theta]$.

8. Show that

   a. the theory of the structure $(\mathbb{C}, +, -, 0, 1)$ of the complex numbers considered as an abelian group with the elements 0 and 1 named has definable Skolem functions while

   b. the theory of the structure $(\mathbb{C}, +, \cdot, -, 0, 1)$ of the complex numbers considered as a field does not have definable Skolem functions.

   [Recall that a theory T in a language $\mathcal{L}$ has definable Skolem functions if for any formula $\psi(x_1, \ldots, x_n, y)$ in the free variables $x_1, \ldots, x_n, y$ there is a definable function $f_\psi(x_1, \ldots, x_n)$ taking the free variables $x_1, \ldots, x_n$ so that $T \vdash (\forall x_1) \cdots (\forall x_n)[\psi(x_1, \ldots, x_n, f_\psi(x_1, \ldots, x_n) \leftrightarrow (\exists y)\psi(x_1, \ldots, x_n, y)]$.]