GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE
FOUNDATIONS PRELIMINARY EXAMINATION

There are eight questions. Partial credit may be assigned for substantially correct partially worked solutions. To pass, you need a score of roughly fifty percent, though the ultimate decision about the passing mark will be decided by the committee of graders.

1. Find a complete theory T in a countable language $\mathcal{L}$ which has exactly $\aleph_0$ countable models up to isomorphism and **prove** that your theory has this property.

2. Show that the following properties of a real number a are equivalent.

(1) The set of rational numbers greater than a is Σ^0_2.
(2) There is a recursive sequence of rational numbers $(r_n : n \in \omega)$ such that $\limsup_{n\to\infty} r_n = a$.

3. Let $\mathcal{L} = \mathcal{L}(E)$ be the language with one binary relation E and no function or constant symbols. Let $\mathfrak{A}$ be the $\mathcal{L}$-structure in which E is interpreted as an equivalence relation all of whose classes are finite and which has one class of size n for each positive integer n. Let $\mathfrak{B}$ be the extension of $\mathfrak{A}$ obtained by adding infinitely new elements to the universe of $\mathfrak{A}$, making them equivalent to each other and inequivalent to the elements of $\mathfrak{A}$. **Prove** $\mathfrak{A} \preceq \mathfrak{B}$.

4.

(a) Show that there is an infinite recursive subtree T of $2^{<\omega}$ such that T has no recursive, infinite branch.
(b) Let T be as in part (a),and let $\theta(v)$ be a formula that defines over the standard model of Peano Arithmetic (PA) the set $I = \{n \mid n \text{ is the code of a sequence that has infinitely many extensions in } T\}$. Show that for some $n \in I$, PA does not prove $\theta(\underline{n})$.

5. Let τ be a finite relational signature. Suppose that $\mathfrak{A}$ is an $\mathcal{L}(\tau)$-structure having the property that for any two finite substructures $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{A}$ and isomorphisms $f : \mathfrak{B} \to \mathfrak{C}$ there is an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{A})$ with $f = \sigma \restriction B$. **Show** that $\mathrm{Th}(\mathfrak{A})$ eliminates quantifiers.

6. Show that the set of indices for recursive convergent sequences of rational numbers is Π^0_3-complete.

7. Let $\mathcal{L} = \mathcal{L}(E)$ be the language with a single binary relation symbol E. Let G and H be two (undirected) graphs regarded $\mathcal{L}$-structures in which E is interpreted as the edge relation. Let f be an elementary embedding from a graph G to H.

(1) **Show** that if H is connected, so is G. (Recall that a graph is connected if any two vertices are connected by a finite path.)
(2) **Give** an example where G is connected but H is not and **prove** that your solution works.

8. Let $\mathcal{L}$ be the language of Peano Arithmetic (PA). For U an axiomatizable theory in $\mathcal{L}$, let $\mathrm{Prov}_U(v_0, v_1)$ naturally represent the relation "x is (the Gödel number of) a proof in U of (the sentence with Gödel number) y", and let Con_U be the associated sentence asserting that U is consistent. Let T be the theory obtained by adding to PA all instances of the schema

$$\exists v \mathrm{Prov}_{\mathsf{PA}}(v, \underline{\theta}) \to \theta,$$

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for θ a sentence in $\mathcal{L}$. (Here $\underline{\theta}$ is the numeral of the Gödel number of θ.) Let $S = \mathsf{PA} + \mathrm{Con}_{\mathsf{PA}}$. Show that $\mathrm{Con}_S \to \mathrm{Con}_T$ is not provable in T.