# Logic and the Methodology of Science 2021 Preliminary Exam

1. Let $L = \{\leq, E\}$, where $E$ is a binary relation. Let $T$ be the $L$-theory saying that $\leq$ is a dense linear order without endpoints, $E$ is an equivalence relation with infinitely many classes, all of which are infinite and each $E$-class is convex (that is if $E(a,b)$ and $a \leq c \leq b$, then $E(a,c)$). This theory admits elimination of quantifiers in $L$ (you do not need to prove this).

   Let $M \models T$ and let $(a_i : i < \omega)$ be an indiscernible sequence of elements of $M$. Assume that $b, c$ are such that $(b) + (a_i : i < \omega)$ and $(a_i : i < \omega) + (c)$ are indiscernible sequences. Show that $(b) + (a_i : i < \omega) + (c)$ is indiscernible.

2. Let $G$ be a connected $\omega$-categorical graph. Show that there exists a natural number $n > 0$ such that any two elements are connect by a path of size at most $n$.

   (A *graph* is a structure that contains only one binary relation $E$ that is symmetric and anti-reflexive. A graph is *connected* if any two elements are connected by a path, that is $\forall x, y \exists z_1, ..., z_k (xEz_1 \wedge z_1Ez_2 \wedge ... \wedge z_kEy)$.)

3. Show that if a theory $T$ is $\aleph_1$-categorical in a countable language, then every uncountable model of $T$ is $\omega$-saturated.

   (Hint: You may start by showing that the model of size $\aleph_1$ is homogeneous and contains a copy of every countable model of $T$.)

4. Let $T$ be theory of $(\mathbb{Q}; 0, 1, +, \cdot)$. Show that $T$ does not admit quantifier elimination in this language.

5. A sentence $\phi$ in the language of arithmetic is called *red* if it is $\Sigma_1$ and for any $\Sigma_1$ sentence $\psi$, $PA \cup \{\phi \leftrightarrow \psi\}$ is consistent.

Let $\phi$ be a $\Sigma_1$ sentence. Show that $\phi$ is red if and only if $\mathbb{N} \models \neg\phi$, but PA does not prove $\neg\phi$.

6. Let $A \subseteq \mathbb{N}$ be an infinite r.e. set. Prove that $\{e : W_e = A\}$ is $\Pi_2$-complete.

7. Show that there is no partial computable function $f$ such that whenever $W_a \neq \emptyset$, $f(a) \in W_a$ and if $W_a = W_b$ are non-empty, then $f(a) = f(b)$. (Hint: you may consider $W_a = \{0, 1\}$, assume $f(a) = 0$.)

8. Show that if $A$ is a $\Pi^0_1$ set of natural numbers, then there is a recursive binary tree $T \subseteq 2^{<\omega}$ such that the set of branches of $T$ is exactly the set of characteristic functions of subsets of $A$.