Preliminary Examination
Group in Logic and the Methodology of Science
September 17, 2018

1. Suppose that φ and ψ are formulas in propositional logic, φ is not a contradiction, ψ is not a tautology, and $(\varphi \to \psi)$ is a tautology. Show that there is a propositional formula θ such that every propositional symbol that appears in θ also appears in both φ and ψ and such that $(\varphi \to \theta)$ and $(\theta \to \psi)$ are tautologies.

2. Show that $(\mathbb{Z}, +)$ is minimal, but not prime.

3. Assume that $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent structures for a countable language. Assume that $\mathfrak{M}$ is countable. Show that $\mathfrak{M}$ embeds into any ultrapower of $\mathfrak{N}$ by a nonprincipal ultrafilter.

4. I is an index set if and only if $e \in I$ and $W_e = W_n$ implies $n \in I$, where $(W_i : i \in \mathbb{N})$ is the canonical indexing of recursively enumerable sets. Let I be a recursively enumerable index set, and let $C = \{W_e : e \in I\}$. Show:

   (a) If $W_e \in C$ then there is a finite subset F of W_e such that $F \in C$.
   (b) If $F \in C$ and $F \subseteq W_n$ then $W_n \in C$.
   (c) There is a recursively enumerable set S of canonical finite sets such that for all e, $e \in I$ if and only if for some $F \in S$, $F \subseteq W_e$.

5. Show that the index set $\{e : W_e \text{ is infinite}\}$ is not recursive relative to $\emptyset'$.

6. Let φ be a Π^0_1 sentence st $PA \vdash (\varphi \to Con(PA))$. Show there is a sentence ψ such that $PA \vdash (\varphi \leftrightarrow Con(PA \cup \{\psi\}))$.

   Hint: for $\varphi = \forall w \theta(w)$, where θ is limited, show there is a Σ^0_1 sentence ψ such that $PA \vdash (\psi \leftrightarrow (\exists p)(\forall w < p)[\theta(w) \land \text{“}p \text{ is a proof of } \neg\psi \text{ from } PA\text{”}])$

7. Let M be a structure and $a \in M$ an element. Assume that there are infinitely many elements in M which have the same type as a (over $\emptyset$). Show that there is an elementary extension $M \prec N$ and an automorphism σ of N such that $a, \sigma(a), \sigma^2(a), \sigma^3(a), \ldots$ are pairwise distinct.

8. Let L be a finite relational language and M an L-structure. Assume that M admits elimination of quantifiers in L. Show that the theory of M is ω-categorical.