Absolutely ordinal definable sets

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References:

(1) Gödel's program, in *Interpreting Gödel*, Juliette Kennedy ed., Cambridge Univ. Press 2014.

(2) Normalizing iteration trees and comparing iteration strategies, to appear.

Plan:

I. Introduction.
II. The consistency-strength hierarchy
III. A theory of the concrete.
IV. A boundary.
V. The multiverse language.
VI. A core for the multiverse?

I. Introduction

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(1) All mathematical language can be translated into LST.
(2) Not all mathematically interesting statements are decided by ZFC.

LST is semantically complete, but ZFC is proof-theoretically incomplete.

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(1) Gödel was concerned mainly with the Continuum Hypothesis (CH).

(2) How does one justify statements in LST? General philosophical questions concerning meaning, evidence, and belief arise.

(3) For those who believe the truth value of CH is not determined by the meaning we currently assign to the syntax of LST, the Continuum Problem does not disappear. Certainly we don't want to employ a syntax which encourages us to ask psuedo-questions, and the problem then becomes how to flesh out the current meaning, or trim back the current syntax, so that we can stop asking psuedo-questions.

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maximize interpretative power.

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Language and theory evolve together.

The consistency strength hierarchy

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Definition

Let T and U be axiomatized theories extending ZFC; then $T \leq_{\mathrm{Con}} U$ iff ZFC proves $\mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$. If $T \leq_{\mathrm{Con}} U$ and $U \leq_{\mathrm{Con}} T$, then we write $T \equiv_{\mathrm{Con}} U$, and say that T and U have the same consistency strength, or are *equiconsistent*.

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These days, the way a set theorist convinces people that T is consistent is to show by forcing that $T \leq_{\mathrm{Con}} H$ for some large cardinal hypothesis H.

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Natural consistency strengths wellordered: If T is a natural extension of ZFC, then there is an extension H axiomatized by large cardinal hypotheses such that $T \equiv_{\mathrm{Con}} H$. Moreover, $\leq_{\mathrm{Con}}$ is a prewellorder of the natural extensions of ZFC. In particular, if T and U are natural extensions of ZFC, then either $T \leq_{\mathrm{Con}} U$ or $U \leq_{\mathrm{Con}} T$.

III. A theory of the concrete

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Definition

Let Γ be a set of sentences in the syntax of LST, and T a theory; then $\Gamma_T = \{\varphi \mid \varphi \in \Gamma \wedge T \vdash \varphi\}$.

A theory of the natural numbers:

Phenomenon: If T and U are natural extensions of ZFC, then

$$\begin{aligned} T \leq_{\mathrm{Con}} U &\Leftrightarrow (\Pi^0_1)_T \subseteq (\Pi^0_1)_U \\ &\Leftrightarrow (\Pi^0_\omega)_T \subseteq (\Pi^0_\omega)_U \end{aligned}$$

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Thus the wellordering of natural consistency strengths corresponds to a wellordering by inclusion of theories of the natural numbers. There is no divergence at the arithmetic level, if one climbs the consistency strength hierarchy in any natural way we know of.

A theory of the reals:

Phenomenon: Let T, U be natural theories of consistency

strength at least that of "there are infinitely many Woodin cardinals"; then either $(\Pi^1_\omega)_T \subseteq (\Pi^1_\omega)_U$, or $(\Pi^1_\omega)_U \subseteq (\Pi^1_\omega)_T$.

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In other words, the second-order arithmetic generated by natural theories is an eventually monotonically increasing function of their consistency strengths.

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Our model-producing methods lead to eventual Γ-monotonicity because in order to produce a model for a theory T that is sufficiently strong with respect to Γ, we must produce a Γ-correct model.

IV. The Levy-Solovay boundary

None of our current large cardinal axioms decide CH, because they are preserved by small forcing, whilst CH can both be made true and made false by small forcing. Because CH is provably not generically absolute, it cannot be decided by large cardinal hypotheses that are themselves generically absolute.

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Theorem (Levy, Solovay)

Let A be one of the current large cardinal axioms, and suppose $V \models A$; then there are set generic extensions M and N of V which satisfy $A + \mathrm{CH}$ and $A + \neg\mathrm{CH}$ repectively.

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CH, is a Σ^2_1 statement. It is the simplest sort of statement large cardinals do not decide. There are many more of them in general set theory.

V. The multiverse language

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We have good evidence that the consistency hierarchy is not a mirage, that the theories in it we have identified are indeed consistent. This argues for developing the theories in this hierarchy. All their Π^0_1 consequences are true, and we know of no other way to produce new Π^0_1 truths.

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This might suggest that we need no further framework: why not simply develop all the natural theories in our hierarchy as tools for generating true statements about the concrete? Let 1000 flowers bloom! This is Hilbertism without the consistency proof, and with perhaps an enlarged class of "real" statements.

Unify!

The problem with this watered-down Hilbertism is that we don't want everyone to have his own private mathematics. We want one framework theory, to be used by all, so that we can use each other's work. It's better for all our flowers to bloom in the same garden. If truly distinct frameworks emerged, the first order of business would be to unify them.

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In fact, the different natural theories we have found in our hierarchy are not independent of one another. Their common theory of the concrete stems from logical relationships that go deeper, and are brought out in our relative consistency proofs. These logical relationships may suggest a unifying framework.

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Maximizing interpretative power entails maximizing consistency strength, but it requires more, in that we want to be able to translate other theories/languages into our framework theory/language in such a way that we preserve their meaning. The way we interpret set theories today is to think of them as theories of inner models of generic extensions of models satisfying some large cardinal hypothesis, and this method has had amazing success. We don't seem to lose any meaning this way. It is natural then to build on this approach.

Beyond large cardinals?

Nevertheless, large cardinal hypotheses like our current ones cannot decide CH, and so our theory of the concrete still has many different possible theoretical superstructures, some with CH, some with $\diamondsuit$, some with MM, some with $2^{\aleph_0}$ being real-vaulued measurable, and so on: all the behaviors that can hold in set-generic extensions of V, no matter what large cardinals exist.

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Before we try to decide whether some such theory is preferable to the others, let us try to find a neutral common ground on which to compare them. We seek a language in which all these theories can be unified, without bias toward any, in a way that exhibits their logical relationships, and shows clearly how they can be used together. That is, we want one neat package they all fit into.

Our neutral common ground.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC.)

(2) (a) Every world is a transitive proper class. An object is a set just in case it belongs to some world.

(b) If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.

(c) If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world.

(d) (Amalgamation.) If U and W are worlds, then there are G, H set generic over them such that $W[G] = U[H]$.

The natural way to get a model of MV is as follows.

Let M be a transitive model of ZFC, and let G be M-generic for $\mathrm{Col}(\omega, < \mathsf{OR}^M)$. The worlds of the multiverse M^G are all those W such that

$$W[H] = M[G \upharpoonright \alpha],$$

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It follows from a result of Laver that the full first order theory of M^G is independent of G, and present in M, uniformly over all M.

That is, there is a recursive translation function t such that whenever M is a model of ZFC and G is $\mathrm{Col}(\omega, < \mathsf{OR}^M)$-generic over M, then

$$M^G \models \varphi \Leftrightarrow M \models t(\varphi),$$

for all sentences φ of the multiverse language. $t(\varphi)$ just says "φ is true in some (equvalently all) multiverse(s) obtained from me".

If $\mathcal{W}$ is a model of MV, then for any world $M \in \mathcal{W}$, there is a G such that $\mathcal{W} = M^G$. Thus assuming MV indicates then that we are using the multiverse language as a sublanguage of the standard one, in the way described above. Also, it is clear that if φ is any sentence in the multiverse language, then MV proves

$$\varphi \Leftrightarrow \text{ for all worlds } M,\ t(\varphi)^M \ \Leftrightarrow \ \text{for some world } M,\ t(\varphi)^M.$$

Thus everything that can be said in the multiverse language can be said using just one world-quantifier.

One can add large cardinal hypotheses that are preserved by small forcing to MV as follows: given such a large cardinal hypothesis φ, we add "φ^W, for all worlds W" to MV.

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There is no obvious way to state CH in the multiverse language.

Have we lost expressive power?

One can think of the standard language as the multiverse language, together with a constant symbol $\dot{V}$ for a reference universe. Statements like CH are intended as statements about the reference universe. To what extent is this constant symbol meaningful? Does one lose anything by retreating to the superficially less expressive multiverse language? We distinguish three answers to this question:

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Weak relativist thesis: Every proposition that can be expressed in the standard language LST can be expressed in the multiverse language.

Strong absolutist thesis: "$\dot{V}$" makes sense, and that sense is not expressible in the multiverse language.

Finally, perhaps weak relativism and the absolutist's idea of a distinguished reference world can be combined, in that that there is an individual world that is definable in the multiverse language.

An elementary forcing argument shows that if the multiverse has a definable world, then it has a unique definable world, and this world is included in all the others. (An observation due to Woodin.) In this case, we call this unique world included in all others the *core* of the multiverse.

Weak absolutist thesis: There are individual worlds that are definable in the multiverse language; that is, the multiverse has a core.

Why weak relativism?

The strongest evidence for the weak relativist thesis is that the mathematical theory based on large cardinal hypotheses that we have produced to date can be naturally expressed in the multiverse sublanguage.

Perhaps we lose something when we do that, some future mathematics built around an understanding of the symbol $\dot{V}$ that does not involve defining $\dot{V}$ in the multiverse language. But at the moment, it's hard to see what that is.

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The weak relativist thesis can be considered as a piece of advice: don't go looking for it.

VI. Does the multiverse have a core?

Whatever one thinks of the semantic completeness of the multiverse language, it does bring the weak absolutist thesis to the fore, as a fundamental question. Because the multiverse language is a sublanguage of the standard one, this is a question for everyone. If the multiverse has a core, then surely it is important, whether it is the denotation of the absolutist's $\dot{V}$ or not!

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Neither MV nor its extensions by large cardinal hypotheses up to the level of supercompact cardinals decides whether there is a core to the multiverse, or the basic theory of this core if it exists (Fuchs, Hamkins, Reitz). But

Theorem

(Usuba 2016) If there is a hyper-huge cardinal, then the generic multiverse V^G has a core.

The Fuchs-Hamkins-Reitz work shows that nothing follows from hyper-huge cardinals concerning the basic theory of the core.

Is the core a canonical inner model?

The canonical inner model M_H for a large cardinal hypothesis H is its most concrete realization. Its construction yields a thorough *fine structure theory* for the model. We have constructed M_H for many H. Do they have a general form?

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The sets in any M_H are ordinal definable in a certain generically absolute way.

Definition

Let $A \subseteq \omega^\omega$; then A is *homogeneously Suslin* (Hom_∞) iff for all κ, there is a system $\langle M_s, i_{s,t} \mid s, t \in \omega^{<\omega} \rangle$ such that

(1) $M_\emptyset = V$, and each M_s is closed under κ-sequences,

(2) for $s \subseteq t$, $i_{s,t} \colon M_s \to M_t$,

(3) if $s \subseteq t \subseteq u$, then $i_{s,u} = i_{t,u} \circ i_{s,t}$, and

(4) for all x, $x \in A$ iff $\lim_n M_{x \restriction n}$ is wellfounded.

Theorem

(Martin, S., Woodin 1985) Assume there are arbitrarily large Woodin cardinals; then for any $A \in Hom_\infty$, $L(A, \mathbb{R}) \models \mathrm{AD}$.

Theorem

(Woodin 1987?) If there are arbitarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ truth is generically absolute.

Remark. CH is Σ^2_1, but definitely not $(\Sigma^2_1)^{Hom_\infty}$. The existential quantifier in CH ranges over wellorders of $\mathbb{R}$, and these cannot be Hom_∞.

V looks like the HOD of a determinacy model

Recall that a set is *ordinal definable* (OD) iff it is definable over the universe of sets from ordinal parameters, and is *hereditarily ordinal definable* (HOD) just in case it and all members of its transitive closure are OD. Gödel first isolated HOD in the 1940s. Myhill and Scott showed that if $M \models \mathrm{ZF}$, then $\mathrm{HOD}^M \models \mathrm{ZFC}$.

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Definition

(Woodin) $V = \textit{ultimate } L$ is the statement: There are arbitrarily large Woodin cardinals, and for any Σ_2 sentence φ of LST: if if φ is true, then for some $A \in Hom_\infty$, $\text{HOD}^{L(A,\mathbb{R})} \models \varphi$.

Theorem

(Woodin) If $V = \textit{ultimate } L$, then

1. *V is the core of its multiverse V^G.*
2. *$V \subseteq HOD^W$ for all W in its multiverse, and $V = HOD^W$ for cofinally many such W.*

One can state the axiom in the multiverse sublanguage: the multiverse has a core, and it satisfies $V = \textit{ultimate } L$.

Ultimate? Like L?

The hope is that $V =$ ultimate L is consistent with all the large cardinal hypotheses, so that adopting it does not restrict interpretative power. It is known to be consistent with the existence of Woodin cardinals. Whether it is consistent with significantly stronger large cardinal hypotheses is a crucial open problem.

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At the same time, one hopes that $V =$ ultimate L will yield a detailed fine structure theory for V, removing the incompleteness that large cardinal hypotheses by themselves can never remove. It is known that $V =$ ultimate L implies the CH, and many instances of the GCH. Whether it implies the full GCH is a crucial open problem.

Theorem

(S. 2015) Suppose there are arbitrarily large Woodin cardinals, and that there is a supercompact cardinal. Assume also that V is uniquely iterable*; then there is a Wadge cut Γ in Hom_∞ such that*

(a) $HOD^{L(\Gamma,\mathbb{R})} \models$ *"there is a superstrong cardinal", and*

(b) $HOD^{L(\Gamma,\mathbb{R})}$ *has a fine structure like that of L; for example,* $HOD^{L(\Gamma,\mathbb{R})} \models \mathrm{GCH}$.

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Big open problems:

(1) Can one remove the iterability hypothesis?

(2) Can one replace "superstrong" by "supercompact" in the conclusion?

Recapitulation

Our current understanding of the possibilities for maximizing interpretative power has led us to to one theory of the concrete, and a family of theoretical superstructures for it, each containing all the large cardinal hypotheses. These different theories are logically related in a way that enables us to use them all together.

Recapitulation

Our current understanding of the possibilities for maximizing interpretative power has led us to to one theory of the concrete, and a family of theoretical superstructures for it, each containing all the large cardinal hypotheses. These different theories are logically related in a way that enables us to use them all together.

The logical relationships between these theories are brought out by formalizing them in the multiverse language. This language is a sublanguage of the standard one, and in it we can formalize naturally all the mathematics that set theorists have done. Remaining within this sublanguage has the additional virtue that our attention is directed away from CH, which has no obvious formalization within it, and toward the global question as to whether the multiverse has a core.

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Thank You!