Model Theory of $\mathbb{T}$

(joint work with Aschenbrenner and van der Hoeven)

$\mathbb{T}$ = ordered differential field of transseries

Goal: to analyse the logical and model-theoretic properties of $\mathbb{T}$, as was achieved by Tarski, A. Robinson, Ax & Kochen, Ercov, ... for the classical fields $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{C}((t))$, ...

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1. Introduction to $\mathbb{T}$
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① Introduction to $\mathbb{T}$

The elements of $\mathbb{T}$ are series like

$$e^{e^x} + 2e^{x\log x} - x^7 + 3 + \frac{1}{x} + \frac{1}{x\log x} + \frac{1}{x(\log x)^2} + \cdots\cdots$$

$$+ \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots\cdots$$

$$+ e^{-x} + 2e^{-x^2} + 3e^{-x^3} + 4e^{-x^4} + \cdots$$

$\mathbb{T}$ has a somewhat lengthy inductive definition (see for example vdD-Macintyre-Marker, APAL 111 (2001))

It has an important subfield $\mathbb{T}_{log}$ consisting of the <u>logarithmic</u> transseries, and this subfield is easy to describe explicitly :

$\ell_0 := x$, $\ell_1 := \log x$, $\cdots$, $\ell_n = \underbrace{\log \cdots \log}_{n} x$

$\mathbb{T}_{\log} = \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \ell_1^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$ where

$\mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$ consists of series $\sum_{(r_0,\dots,r_n)} c_{(r_0,\dots,r_n)} \ell_0^{r_0} \cdots \ell_n^{r_n}$

NB: $\ell_0^{-1} + \ell_0^{-1}\ell_1^{-1} + \cdots + \ell_0^{-1}\ell_1^{-1}\cdots\ell_n^{-1} + \cdots \notin \mathbb{T}$

but defines an important cut in $\mathbb{T}$: it is a key feature of $\mathbb{T}$ that it is <u>not</u> spherically complete

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$\mathbb{T}$ is an ordered field extension of $\mathbb{R}$ with $x > \mathbb{R}$, and $\mathbb{T}$ is an <u>exponential</u> field:

exp(series on previous page)

$$=$$

$$\underbrace{\exp(\text{infinite part})}_{\text{new monomial}} \cdot e^3 \cdot \underbrace{\exp(\text{infinitesimal part})}_{\text{to be expanded according to}}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\text{infinitesimal part})^n$$

Marker - Macintyre - vdD : $(\mathbb{R}, \exp) \prec (\mathbb{T}, \exp)$ (1990's), but here we focus on something else:

in $\mathbb{T}$ we can differentiate termwise:

$r' = 0$ for $r \in \mathbb{R}$, $x' = 1$, $(e^f)' = f'e^f$

$$\boxed{\text{From now on we view } \mathbb{T} \text{ as an ordered differential field}}$$

$\therefore$ $\mathbb{R} = \{f \in \mathbb{T} : f' = 0\}$

is a definable (discrete) subfield of $\mathbb{T}$

<u>Écalle</u>: It seems that $\mathbb{T}$ is truly the algebra-from-which-one-can-never-exit and that marks an almost impassable horizon for "ordered analysis"

(2) Conjectures about $\mathbb{T}$

Initial Conjecture: $\boxed{\mathbb{T} \text{ is model complete}}$

In algebraic terms, this means the following:

Define a D-algebraic set in $\mathbb{T}^n$ to be a set

$$\{f \in \mathbb{T}^n : P_1(f) = \cdots = P_k(f) = 0\} \quad \text{for some}$$

D-polynomials $P_i(y_1, \ldots, y_n, y_1', \ldots, y_n', y_1'', \ldots, y_n'', \ldots)$ over $\mathbb{T}$. Call a set $S \subseteq \mathbb{T}^m$ sub-D-algebraic if it is the image of a D-algebraic set in $\mathbb{T}^n$ under the natural projection map $\mathbb{T}^n \to \mathbb{T}^m$ for some $n \geq m$.

Model completeness means (a little more than): the complement of every sub-D-algebraic set in $\mathbb{T}^m$ is sub-D-algebraic ($\therefore$ every definable set in every $\mathbb{T}^m$ is sub-D-algebraic)

More precise conjecture:

$\boxed{\text{Th}(\mathbb{T}) \text{ is the model companion of the theory of H-fields}}$

H : Hardy, Hausdorff
Hahn, Borel

H-fields are the ordered differential fields K such that $\mathcal{O} = C + \mathcal{o}$ where

$$C = \{a \in K : a' = 0\} \quad \text{(field of constants of } K)$$

$$\mathcal{O} = \{a \in K : |a| \leq c \text{ for some } c \in C\}$$

$$\mathcal{o} = \{a \in K : |a| < c \text{ for all positive } c \in C\}$$

and such that for all $a \in K$

$$a \in \mathcal{o} \implies a' \in \mathcal{o}$$

$$a > C \implies a' > 0.$$

Examples: $\mathbb{T}$, every Hardy field containing $\mathbb{R}$ such as $\mathbb{R}(x)$, $\mathbb{R}(x, \log x)$, ...

The "model companion" conjecture adds to the model completeness conjecture that H-fields are exactly the ordered differential fields embeddable into ultrapowers of $\mathbb{T}$

This suggests an approach:
study the extension theory of H-fields.
It helps to know that H-fields fall under the so-called "differential-valued fields" of Rosenlicht ($\pm$ 1980)

Consequence of model companion conjecture:
Differential Nullstellensatz for $\mathbb{T}$

Beyond being an H-field, $\mathbb{T}$ is Liouville closed

An H-field is Liouville closed

$$\overset{\text{def}}{\Longleftrightarrow}$$

real closed and $\forall a \exists b (a = b')$, $\forall a \exists b_{\neq 0} (a = b^\dagger)$

$$b^\dagger := b'/b$$

Every H-field has a minimal Liouville closed extension. The trouble is that it can have two!

$\therefore$ don't expect QE for $Th(\mathbb{T})$ in the natural language of ordered differential fields, or even in the language of ordered valued differential fields

Related Conjectures

- $\mathbb{T}$ is o-minimal at ∞: if $X \subseteq \mathbb{T}$ is definable, then $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$ for some $f \in \mathbb{T}$
- all subsets of $\mathbb{R}^n$ definable in $\mathbb{T}$ are semialgebraic
- $\mathbb{T}$ has NIP

All conjectures above do indeed follow from our recent QE for $\mathbb{T}$ in the language of ordered valued differential fields augmented by unary predicates Λ and Ω:

$$\Lambda(a) \iff a < \ell_0^{-1} + \ell_0^{-1}\ell_1^{-1} + \cdots + \ell_0^{-1}\ell_1^{-1}\cdots\ell_n^{-1} \text{ for some } n$$

$$\Omega(a) \iff a < \ell_0^{-2} + \ell_0^{-2}\ell_1^{-2} + \cdots + \ell_0^{-2}\ell_1^{-2}\cdots\ell_n^{-2} \text{ for some } n.$$

$$\iff 4y'' + ay = 0 \text{ for some } y \neq 0$$

③ Early (<2000) Evidence

(i) The asymptotic couple of $\mathbb{T}$ has good model theoretic properties; its theory is the model companion of the theory of H-asymptotic couples

(ii) Intermediate Values: for any D-polynomial $P(y)$ over $\mathbb{T}$ and $a, b \in \mathbb{T}$, $a < b$, $P(f)$ takes all values in $\mathbb{T}$ between $P(a)$ and $P(b)$ for $a \leq f \leq b$.

④ Present state of knowledge

The following axiomatizes a complete theory:

- Liouville closed H-field
- ω-free (to be explained)
- newtonian (" " ")

Moreover, $\mathbb{T}$ is a model of these axioms (For "newtonian" this relies on published work by van der Hoeven, not yet verified by Aschenbrenner and me.)

This theory has QE in the language of ordered valued differential fields augmented by unary predicates Λ and Ω with defining axioms:

$$\Lambda(f) \longleftrightarrow \exists y\,[y > C \;\&\; f = -y^{\dagger\dagger}]$$

$$\Omega(f) \longleftrightarrow \exists y\,[4y'' + fy = 0]$$

What is "ω-free"?

Let K be an H-field with asymptotic integration, that is, $K \models \forall a_{\neq 0}\ \exists b\,[v(a-b') > v(a)]$,

where v is the valuation on K with valuation ring $\mathcal{O}$ = convex hull of C

Such K is ω-free

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$K \models \forall a\, \exists b\,[v(b) < 0,\ 2v(b) \geq v(a - 2(b^{++})' - (b^{++})^2)]$$

This is also equivalent to K having just one minimal Liouville closed extension

ω-freeness is amazingly robust, for example, every D-algebraic H-field extension of an ω-free H-field is still ω-free

What is "newtonian" ?

It's a bit like (differential) henselian, and says that certain kinds of differential polynomials in one variable have a zero in $\mathcal{O}$.

It can be expressed as a first-order axiom scheme. Recent results:

- Any ω-free H-field with divisible value group has a unique newtonization
- Any ω-free H-field ~~with divisible value group~~ has a unique Newton-Liouville closure
- If K is an ω-free newtonian Liouville closed H-field, then K has no proper D-algebraic H-field extension with the same constant field

Things to do

- construct Hardy field models;
  what about maximal Hardy fields

- what about $\mathbb{T}_{\log}$? It is newtonian.
  Also encouraging: its asymptotic couple is alright (Allen Gehret)

- what about $\mathbb{T}$ as a differential exponential field?

- connection to Conway's surreals?