Conversational dynamics: technical notes

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[The material below is from our paper ‘On the dynamics of conversation’.]

1 Conversation systems

Def 1. A CONVERSATION SYSTEM is a triple \( \langle L, C, \cdot \rangle \), where \( L \) is a set of sentences, \( C \) is a set of informational contexts, and \( \cdot \) is an update function from \( L \) to a set of context-change potentials (unary operations) on \( C \).

Def 2. A PROPOSITION MAP is a triple \( \langle L, P, \llbracket \cdot \rrbracket \rangle \), where \( L \) is a set of sentences, \( P \) is a set of propositions, and \( \llbracket \cdot \rrbracket \) is a mapping with \( \llbracket \cdot \rrbracket : L \to P \).

Def 3. A conversation system \( \langle L, C, \cdot \rangle \) is INCREMENTAL if and only if there exists a proposition map \( \langle L, P, \llbracket \cdot \rrbracket \rangle \) and a one-to-one function \( f \) from \( C \) to \( P(P) \) such that for all \( c \in C \) and \( s \in L \), \( f(c) \cup \llbracket \lnot s \rrbracket = f(c[s]) \).

Def 4. A conversation system \( \langle L, C, \cdot \rangle \) is STATIC if and only if there exists a set of sets \( P \), a proposition map \( \langle L, P, \llbracket \cdot \rrbracket \rangle \), and a one-to-one function \( f \) from \( C \) to \( P \) such that for all \( c \in C \) and \( s \in L \), \( f(c) \cap \llbracket s \rrbracket = f(c[s]) \).

2 van Benthem staticness

Def 5. A conversation system \( \langle L, B, \cdot \rangle \) is VAN BENThEM STATIC iff there exists a Boolean algebra\(^2\) \( B_A \), \( B_A = \langle B, \land, \lor, \lnot, \top, \bot \rangle \), such that for all \( c \in B \) and \( s \in L \),

\[ c[s] \lor c = c \]

\(^1\)Conversational system= deterministic labelled state transition system.
\(^2\)A BOOLEAN ALGEBRA is a tuple \( \langle B, \land, \lor, \lnot, \top, \bot \rangle \), where \( B \) is a set, \( \land, \lor \) are binary operations on \( B \), \( \lnot \) is a unary operation on \( B \), and \( \top, \bot \in B \), such that: for any \( x, y \in B \): (1) \( x \lor (x \land y) = x \); (2) \( x \land (x \lor y) = x \); (3) \( x \lor \lnot x = \top \); (4) \( x \land \lnot x = \bot \).

Eliminativity. \( c[s] \lor c = c \)

Finite distributivity. \( (c \lor c')[s] = c[s] \lor c'[s] \)

Call any such triple \( \langle L, B_A, \cdot \rangle \) a van Benthem static conversation system with BOOLEAN STRUCTURE.

Fact 1 (van Benthem 1986). If \( \langle L, B_A, \cdot \rangle \) is a van Benthem static conversation system with Boolean structure, where \( B_A = \langle B, \land, \lor, \lnot, \top, \bot \rangle \), then for all \( c \in B \) and \( s \in L \): \( c[s] = c \land \top[s] \).

Proof. \( c \land \top[s] = c \land (c \lor \lnot c)[s] \)

\[ = c \land (c[s] \lor \lnot c[s]) \quad \text{(Finite distributivity)} \]

\[ = (c \land c[s]) \lor (c \land \lnot c[s]) \]

\[ = c[s] \lor \emptyset \quad \text{(Eliminativity)} \]

Fact 2. If a conversation system is van Benthem static, it is static.

3 Veltman staticness

Def 6. A quadruple \( \langle V, \top, \land, \leq \rangle \) is an INFORMATION LATTICE iff \( V \) is a set, \( \top \in V \), \( \land \) is a binary operation on \( V \), and \( \leq \) is a partial order on \( V \) such that for all \( c, c' \in V \):

\[ \top \land c = c \]

\[ c \land c = c \]

\[ c \land c' = c' \land c \]

\[ (c \land c') \land c'' = c \land (c' \land c'') \]

\[ c \leq c' \text{ iff there is some } c'' \text{ such that } c \land c'' = c'. \]

Def 7. A conversation system \( \langle L, V, \cdot \rangle \) is VELTMAN STATIC iff there exists an information lattice, \( V_I \), \( V_I = \langle V, \top, \land, \leq \rangle \), such that for all \( c, c' \in V \) and \( s \in L \),

Idempotence. \( c[s][s] = c[s] \)

Persistence. If \( c[s] = c \) and \( c \leq c' \) then \( c'[s] = c' \)

Strengthening. \( c \leq c[s] \)

Monotony. If \( c \leq c' \) then \( c[s] \leq c'[s] \)

\(^3\)The specification of \( \leq \) adds no structure as it is induced by \( \land \), but we will find the explicit specification convenient below. An intuitive gloss on \( c \leq c' \) would be “\( c' \) is at least as informationally strong as \( c \)”. 
Call any such triple \( (L, V, [\cdot]) \) a Veltman static update system with information structure.

**Fact 3** (Veltman 1996). If \( (L, V, [\cdot]) \) is a Veltman static conversation system with information structure, where \( V = \{ \top, \land, \leq \} \), then for all \( c \in V \) and \( s \in L \): \( c[s] = c \land \top[s] \).

**Proof.** \( e \leq c \land \top[s] \)
\[ c[s] \leq (e \land \top[s])[s] \quad \text{(Monotony)} \]
\[ c[s] \leq c \land \top[s] \quad \text{(Idempotence, Persistence)} \]
For the other direction:
\[ \top[s] \leq c[s] \quad \text{(Idempotence, Monotony)} \]
\[ c \land \top[s] \leq c[s] \land c \quad \text{(Strengthening)} \]
\[ c \land \top[s] = c[s] \]
\[ \square \]

**Fact 4.** If a conversation system is Veltman static, it is static.

![Figure 1](image1.png) A Veltman static conversation system that is not van Benthem static. The information lattice is \( (V = \{ \emptyset, \{0\}, \{0, 1\}, \name{\top}, \name{\leq} \} \), \( \top = \{0, 1\}, \name{\land}, \name{\leq} \} \). The conversation system is \( (\{a, b\}, V, [\cdot]) \), where for all \( c \in V \), \( c[a] = c \cap \{1\} \) and \( c[b] = \emptyset \).

![Figure 2](image2.png) A static conversation system that is not Veltman static.

## 4 Staticness characterized

**Fact 5** (Static representation theorem). A conversation system \( (L, C, [\cdot]) \) is static iff for all \( s, s' \in L \) and \( c \in C \),

**Idempotence.** \( c[s][s] = c[s] \)

**Commutativity.** \( c[s][s'] = c[s'][s] \)

We begin with the right-to-left direction.

**Fact 5.1** If a conversation system is idempotent and commutative, then it is static.

**Proof.** Let \( (L, C, [\cdot]) \) be an idempotent and commutative conversation system. To show that the system is static, it will suffice to show that there exists a proposition map \( (L, \mathcal{P}(C), [\cdot]) \) and an injective function \( f : C \to \mathcal{P}(C) \) such that \( f(c[s]) = f(c) \cap [s] \), for all \( s \in L \) and \( c \in C \).

In order to define \( f \) and \([\cdot]\), we first define a relation \( \leq_U \) between contexts in an arbitrary conversation system \( U \), as follows:

**Def 8.** For any conversation system \( U \), and \( c, c' \in C_U \), \( c \leq_U c' \) iff there exist \( s_1, \ldots, s_n \in L_U \) such that \( c[s_1] \ldots [s_n] = c' \), or \( c = c' \). (We will just write \( \leq \) if the conversation system being discussed is clear from context.)

We will find the following abbreviation useful: since \([\cdot]\) is commutative, we can speak of the update of a set of sentences on a context irrespective of their sequential order:

**Def 9.** If \( S \) is a finite set of sentences \( s_1 \ldots s_n \) from \( L \), \( c[S] =_{\text{def}} c[s_1] \ldots [s_n] \).

We pause to observe that relative to any commutative idempotent conversation system, \( \leq \) is transitive, reflexive and anti-symmetric. Reflexivity is trivial. Transitivity: suppose \( c_1 \leq c_2 \) and \( c_2 \leq c_3 \). Then for some \( S, S' \), \( c_1[S] = c_2 \) and \( c_1[S'] = c_3 \); hence \( c_1[S][S'] = c_3 \), so \( c_1 \leq c_3 \). Anti-symmetry: suppose \( c_1 \leq c_2 \) and \( c_2 \leq c_1 \). Then for some \( S, S' \), \( c_1[S] = c_2 \) and \( c_2[S'] = c_1 \), and hence \( c_1[S][S'] = c_1 \). By commutativity it follows that \( c_1[S'][S] = c_1 \), and hence \( c_1[S'][S][S'] = c_1[S] \). By idempotence \( c_1[S'][S][S] = c_1[S][S] \), so substituting, \( c_1[S'][S] = c_1[S] \); substituting again, \( c_1 = c_2 \).

Define \( f : C \to \mathcal{P}(C) \) as follows: \( f(c) =_{\text{def}} \{ c' \in C : c \leq c' \} \). We observe \( f \) is an injection (i.e., if \( f(c_1) = f(c_2) \) then \( c_1 = c_2 \), for all \( c_1, c_2 \in C \).
Suppose \( f(c_1) = f(c_2) \). Now \( f(c_1) = \{ c' \in C : c_1 \leq c' \} \), hence by reflexivity \( c_1 \in f(c_1) \). Hence \( c_1 \in f(c_2) \); hence \( c_1 \in \{ c' \in C : c_2 \leq c' \} \) and therefore \( c_2 \leq c_1 \). By parity, \( c_2 \in f(c_1) \), and \( c_1 \leq c_2 \). By anti-symmetry, \( c_1 = c_2 \).

Now define \([\underline{\cdot}] : L \rightarrow \mathcal{P}(C)\) to be the minimum function such that \([\underline{\cdot}]\) takes \( s \) to its fixed points on the update function \([\cdot]\).

The preceding defines (i) a proposition map \( \langle L, \mathcal{P}(C), [\underline{\cdot}] \rangle \) given an arbitrary commutative idempotent conversation system \( \langle L, C, [\cdot] \rangle \), and (ii) a injective function \( f \) from \( C \rightarrow \mathcal{P}(C) \). It remains to show that for all \( c \in C \) and \( s \in L \), \( f(c[s]) = f(c) \cap [s] \).

First we show that if \( c_1 \in f(c[s]) \), then \( c_1 \in f(c) \cap [s] \). Suppose \( c_1 \in f(c[s]) \).

(i) Then \( c_1 \in \{ c' \in C : c[s] \leq c' \} \). So \( c[s] \leq c_1 \). By definition \( c \leq c[s] \). So \( c \leq c[s] \leq c_1 \). Hence by transitivity \( c \leq c_1 \), hence \( c_1 \in f(c) \). (ii) Now since \( c[s] \leq c_1 \), there exists some \( S \) such that \( c[s][S] = c_1 \). So \( c[s][S][s] = c_1[s] \).

By commutativity, \( c[s][S][s] = c[S][s][s] \), which by idempotence equals \( c[S][s] \), which by commutivity equals \( c[s][S] \). So \( c[s][S][s] = c[s][S] \).

Here we substitute \( c_1 \) for \( c[s][S] \), and we have \( c_1[s] = c_1 \). From this it follows that \( c_1 \in [s] \), since the latter just is \( \{ c \in C : c[s] = c \} \). So from (i) and (ii) we have \( c_1 \in f(c) \cap [s] \), the desired result.

Now let us show that if \( c_1 \in f(c) \cap [s] \), then \( c_1 \in f(c[s]) \). This is equivalent to showing that if \( c_1[s] = c_1 \) and \( c \leq c_1 \), then \( c_1[s] \leq c_1 \). Suppose \( c \leq c_1 \). Then there is some \( S \) such that \( c[S] = c_1 \). Suppose also \( c_1[s] = c_1 \). Then we have \( c[S] = c_1[s] \). Therefore \( c[s][S] = c_1 \) by commutativity \( c[s][S] = c_1 \). And that means \( c[s] \leq c_1 \); and therefore \( c_1 \in f(c[s]) \).

The left-right direction completes the proof:

**Fact 5.2** If a conversation system is static, then it is commutative and idempotent.

**Proof.** Any static system is idempotent and commutative, since intersection is idempotent and commutative.

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### 5 Commutativity

(1) a. Harry is married. Harry’s spouse is a great cook.
   b. ?Harry’s spouse is a great cook. Harry is married.

(2) a. [A man] walked in. He was tall.
   b. ?He was tall. [A man] walked in.

(3) a. Billy might be at the door.... Billy is not at the door.
   b. ?Billy is not at the door... Billy might be at the door.

![Figure 3: Merely reversing the order of sentences in natural language conversation does not result in commutation.](image)

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### 6 Information-sensitivity characterized

**Def 10.** An INFORMATION-RELATIVE PROPOSITION MAP is a quadruple \( \langle L, C, P, [\underline{\cdot}] \rangle \), where \( L \) is a set of sentences, \( C \) is a set of contexts, \( P \) is a set of propositions, and \( [\underline{\cdot}] \) is a mapping with \([\underline{\cdot}] : L \times C \rightarrow P \).

**Def 11.** A conversation system \( \langle L, C, [\cdot] \rangle \) is INFORMATION-SENSITIVE if and only if there exists a set of sets \( P \), an information-sensitive proposition map \( \langle L, C, P, [\underline{\cdot}] \rangle \), and a one-to-one function \( f \) from \( C \) to \( P \) such that for all \( c \in C \) and \( s \in L \), \( f(c) \cap [s]^c = f(c[s]) \).
Def 12. A conversation system \( \langle L, C, \cdot \rangle \) is MONOTONIC just in case for all \( s_i \in L \) and \( c \in C \), if \( c[s_i] \neq c \), then for all ordered sequences \( s_1 \ldots s_n \) of elements of \( L \), \( c[s_1] \ldots c[s_n] \neq c \).

Then the observation is this:

**Fact 6.** A conversation system is information-sensitive just in case it is monotonic.

*Proof.* Recall the definition of \( \leq_U \) in the proof of Fact 5 above:

**Def 8.** For any conversation system \( U \), and \( c, c' \in \mathcal{C}_U \), \( c \leq_U c' \) iff there exist \( s_1 \ldots s_n \in L_U \) such that \( c[s_1] \ldots c[s_n] = c' \), or \( c = c' \).

Observe that a conversation system \( U = \langle L, C, \cdot \rangle \) is monotonic just in case \( \leq_U \) is a partial order. Thus it suffices to show that \( U \) is information-sensitive iff \( \leq_U \) is a partial order. We first show that if \( U \) is information-sensitive, the corresponding \( \leq \) is a partial order. Reflexivity and transitivity are immediate consequences of the definition of \( \leq \). For anti-symmetry simply note that for \( c, d \in C' \), \( c \leq d \) only if \( c = d \). It follows that if \( c \leq d \) and \( d \leq c \), then \( c = d \).

We now show that if \( c \) is a partial order, then \( U \) is information-sensitive. Let \( f : C \rightarrow \mathcal{P}(C) \) be such that \( f(c) : \{ c' : c \leq c' \} \). Note that \( f \) is injective, as we showed in our proof of Fact 5 using only the fact that \( \leq \) is a partial order. Now consider the information-relative proposition map \( \langle L, C, \mathcal{P}(C), \mathcal{P}(\cdot) \rangle \) where \( [s] \) is the minimal mapping such that for all \( c \in C \) and \( s \in L \), \( [s]^c = f(c[s]) \). It remains to establish that for all \( c \in C \) and \( s \in L \), \( f(c) \cap [s]^c = f(c[s]) \). First, we show that \( [s]^c \subseteq f(c) \). Note that \( [s]^c = f(c[s]) \) and \( c \leq c[s] \). Now, suppose \( s' \in f(c[s]) \), then, by definition, \( c[s] \leq c' \). So, by transitivity of \( \leq \), \( c \leq c' \), and \( s' \in f(c) \). So \( [s]^c \subseteq f(c) \). It follows immediately that: \( f(c) \cap [s]^c = f(c[s]) \).

**Fact 7.** If a conversation system is incremental, then it is static.

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**Appendix**

**Fact 8.** Not every intersective system is incremental.

*Proof.* Consider an intersective conversation system \( \langle L, C, \cdot \rangle \) such that:

1. \( c_1 [p \land q] [p] = c_1 [p \land q] \)
2. \( c_1 [p \land q] \neq c_1 [p] \neq c_1 \).

Suppose the system is incremental. Then there exists some proposition map \( \langle L, P, \mathcal{P}(\cdot) \rangle \) and a one-to-one function \( f \) from \( C \) to \( \mathcal{P}(P) \) such that for all \( c \in C \) and \( s \in L \), \( f(c) \cup \{ [s] \} = f(c[s]) \). Given such a map, we know

\[
\begin{align*}
(f(c_1 [p \land q] [p])) &= f(c_1) \cup \{ [p \land q] \} \cup \{ [p] \} \\
(f(c_1) \cup \{ [p \land q] \} \cup \{ [p] \}) &= f(c_1) \cup \{ [p \land q] \} \cup \{ [p] \}
\end{align*}
\]

Since by (i) we have \( c_1 [p \land q] [p] = c_1 [p \land q] \), it follows that

\[
(f(c_1) \cup \{ [p \land q] \} \cup \{ [p] \}) = f(c_1) \cup \{ [p \land q] \} \cup \{ [p] \}
\]

This entails that either \( [p] \in f(c_1) \) or \( [p] \in \{ [p \land q] \} \) (n.b., \( \{ [p] \} \) is a singleton). Suppose the former. Then \( f(c_1) \cup \{ [p] \} = f(c_1) = f(c_1 [p]) \); but since \( f \) is one-one, this result is incompatible with (ii), which says \( c_1 [p] \neq c_1 \). So suppose instead \( [p] \in \{ [p \land q] \} \). But this entails \( [p] = [p \land q] \), meaning \( f(c_1 [p \land q]) = f(c_1 [p]) \). Since \( f \) is one-one, this result is incompatible with (ii), which says \( c_1 [p \land q] \neq c_1 [p] \).