Conversational dynamics: technical notes

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[The material below is from our paper 'On the dynamics of conversation'.]

1 Conversation systems

Def 1. A CONVERSATION SYSTEM is a triple $\langle L, C, [\cdot]\rangle$, where L is a set of sentences, C is a set of informational contexts, and $[\cdot]$ is an *update function* from L to a set of context-change potentials (unary operations) on C.[1]

Def 2. A PROPOSITION MAP is a triple $\langle L, P, [\![\cdot]\!]\rangle$, where L is a set of sentences, P is a set of propositions, and $[\![\cdot]\!]$ is a mapping with $[\![\cdot]\!] : L \to P$.

Def 3. A conversation system $\langle L, C, [\cdot]\rangle$ is INCREMENTAL if and only if there exists a proposition map $\langle L, P, [\![\cdot]\!]\rangle$ and a one-to-one function f from C to $\mathcal{P}(P)$ such that for all $c \in C$ and $s \in L$, $f(c) \cup \{[\![s]\!]\} = f(c[s])$.

Def 4. A conversation system $\langle L, C, [\cdot]\rangle$ is STATIC if and only if there exists a set of sets P, a proposition map $\langle L, P, [\![\cdot]\!]\rangle$, and a one-to-one function f from C to P such that for all $c \in C$ and $s \in L$, $f(c) \cap [\![s]\!] = f(c[s])$.

2 van Benthem staticness

Def 5. A conversation system $\langle L, B, [\cdot]\rangle$ is VAN BENTHEM STATIC iff there exists a Boolean algebra[2] B_A, $B_A = \langle B, \wedge, \vee, \neg, \top, \bot\rangle$, such that for all $c \in B$ and $s \in L$,

Eliminativity. $c[s] \vee c = c$

Finite distributivity. $(c \vee c')[s] = c[s] \vee c'[s]$

Call any such triple $\langle L, B_A, [\cdot]\rangle$ a van Benthem static conversation system WITH BOOLEAN STRUCTURE.

Fact 1 (van Benthem 1986)**.** If $\langle L, B_A, [\cdot]\rangle$ is a van Benthem static conversation system with Boolean structure, where $B_A = \langle B, \wedge, \vee, \neg, \top, \bot\rangle$, then for all $c \in B$ and $s \in L$: $c[s] = c \wedge \top[s]$.

Proof.
$$
\begin{aligned}
c \wedge \top[s] &= c \wedge (c \vee \neg c)[s] \\
&= c \wedge (c[s] \vee \neg c[s]) && \text{(Finite distributivity)} \\
&= (c \wedge c[s]) \vee (c \wedge \neg c[s]) \\
&= c[s] \vee \emptyset && \text{(Eliminativity)} \\
&= c[s]
\end{aligned}
$$
□

Fact 2. If a conversation system is van Benthem static, it is static.

3 Veltman staticness

Def 6. A quadruple $\langle V, \top, \wedge, \leq\rangle$ is an INFORMATION LATTICE iff V is a set, $\top \in V$, $\wedge$ is a binary operation on V, and $\leq$ is a partial order on V such that for all $c, c' \in V$:

$\top \wedge c = c$
$c \wedge c = c$
$c \wedge c' = c' \wedge c$
$(c \wedge c') \wedge c'' = c \wedge (c' \wedge c'')$
$c \leq c'$ iff there is some c'' such that $c \wedge c'' = c'$.[3]

Def 7. A conversation system $\langle L, V, [\cdot]\rangle$ is VELTMAN STATIC iff there exists an information lattice, V_I, $V_I = \langle V, \top, \wedge, \leq\rangle$, such that for all $c, c' \in V$ and $s \in L$,

Idempotence. $c[s][s] = c[s]$
Persistence. If $c[s] = c$ and $c \leq c'$ then $c'[s] = c'$
Strengthening. $c \leq c[s]$
Monotony. If $c \leq c'$ then $c[s] \leq c'[s]$

[1] Conversation system=deterministic labelled state transition system.

[2] A BOOLEAN ALGEBRA is a tuple $\langle B, \wedge, \vee, \neg, \top, \bot\rangle$, where B is a set, $\wedge, \vee$ are binary operations on B, $\neg$ is a unary operation on B, and $\top, \bot \in B$, such that: for any $x, y \in B$: (1) $x \vee (x \wedge y) = x$; (2) $x \wedge (x \vee y) = x$; (3) $x \vee \neg x = \top$; (4) $x \wedge \neg x = \bot$.

[3] The specification of $\leq$ adds no structure as it is induced by $\wedge$, but we will find the explicit specification convenient below. An intuitive gloss on $c \leq c'$ would be "c' is at least as informationally strong as c".

Call any such triple $\langle L, V_I, [\cdot]\rangle$ a Veltman static update system WITH INFORMATION STRUCTURE.

Fact 3 (Veltman 1996)**.** If $\langle L, V_I, [\cdot]\rangle$ is Veltman static conversation system with information structure, where $V = \langle V, \top, \wedge, \leq\rangle$, then for all $c \in V$ and $s \in L$: $c[s] = c \wedge \top[s]$.

Proof. $c \leq c \wedge \top[s]$
$c[s] \leq (c \wedge \top[s])[s]$ (Monotony)
$c[s] \leq c \wedge \top[s]$ (Idempotence, Persistence)
For the other direction:
$\top \leq c[s]$
$\top[s] \leq c[s]$ (Idempotence, Monotony)
$c \wedge \top[s] \leq c[s] \wedge c$
$c \wedge \top[s] \leq c[s]$ (Strengthening)
$c \wedge \top[s] = c[s]$ □

Fact 4. If a conversation system is Veltman static, it is static.

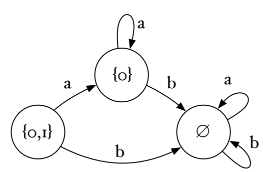

Figure 1: A Veltman static conversation system that is not van Benthem static. The information lattice is $\langle V = \{\emptyset, \{0\}, \{0,1\}\}, \top = \{0,1\}, \cap, \supseteq\rangle$. The conversation system is $\langle\{\text{a,b}\}, V, [\cdot]\rangle$, where for all $c \in V$, $c[\text{a}] = c \cap \{1\}$ and $c[\text{b}] = \emptyset$.

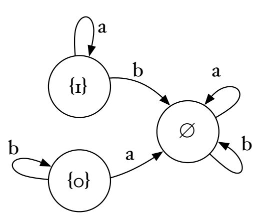

Figure 2: A static conversation system that is not Veltman static.

4 Staticness characterized

Fact 5 (Static representation theorem)**.** A conversation system $\langle L, C, [\cdot]\rangle$ is static iff for all $s, s' \in L$ and $c \in C$,

Idempotence. $c[s][s] = c[s]$
Commutativity. $c[s][s'] = c[s'][s]$

We begin with the right-to-left direction.

Fact 5.1 If a conversation system is idempotent and commutative, then it is static.

Proof. Let $\langle L, C, [\cdot]\rangle$ be an idempotent and commutative conversation system. To show that the system is static, it will suffice to show that there exists a proposition map $\langle L, \mathcal{P}(C), [\![\cdot]\!]\rangle$ and an injective function $f : C \to \mathcal{P}(C)$ such that $f(c[s]) = f(c) \cap [\![s]\!]$, for all $s \in L$ and $c \in C$.

In order to define f and $[\![\cdot]\!]$, we first define a relation $\leq_U$ between contexts in an arbitrary conversation system U, as follows:

Def 8. For any conversation system U, and $c, c' \in C_U$, $c \leq_U c'$ iff there exist $s_1 \ldots s_n \in L_U$ such that $c[s_1]....[s_n] = c'$, or $c = c'$. (We will just write $\leq$ if the conversation system being discussed is clear from context.)

We will find the following abbreviation useful: since $[\cdot]$ is commutative, we can speak of the update of a set of sentences on a context irrespective of their sequential order:

Def 9. If S is a finite set of sentences $s_1....s_n$ from L, $c[S] =_{\text{DEF}} c[s_1]....[s_n]$.

We pause to observe that relative to any commutative idemopotent conversation system, $\leq$ is transitive, reflexive and anti-symmetric. Reflexivity is trivial. Transitivity: suppose $c_1 \leq c_2$ and $c_2 \leq c_3$. Then for some S, S', $c_1[S] = c_2$ and $c_1[S'] = c_3$; hence $c_1[S][S'] = c_3$, so $c_1 \leq c_3$. Anti-symmetry: suppose $c_1 \leq c_2$ and $c_2 \leq c_1$. Then for some S, S', $c_1[S] = c_2$ and $c_2[S'] = c_1$, and hence $c_1[S][S'] = c_1$. By commutativity it follows that $c_1[S'][S] = c_1$, and hence $c_1[S'][S][S] = c_1[S]$. By idempotence $c_1[S'][S][S] = c_1[S'][S]$, so substituting, $c_1[S'][S] = c_1[S]$; substituting again, $c_1 = c_2$.

Define $f : C \to \mathcal{P}(C)$ as follows: $f(c) =_{\text{DEF}} \{c' \in C : c \leq c'\}$. We observe f is an injection (i.e., if $f(c_1) = f(c_2)$ then $c_1 = c_2$, for all $c_1, c_2 \in C$.)

Suppose $f(c_1) = f(c_2)$). Now $f(c_1) = \{c' \in C : c_1 \leq c'\}$, hence by reflexivity $c_1 \in f(c_1)$. Hence $c_1 \in f(c_2)$; hence $c_1 \in \{c' \in C : c_2 \leq c'\}$ and therefore $c_2 \leq c_1$. By parity, $c_2 \in f(c_1)$, and $c_1 \leq c_2$. By anti-symmetry, $c_1 = c_2$.

Now define $[\![\cdot]\!] : L \to \mathcal{P}(C)$ to be the minimum function such that $[\![s]\!] = \{c \in C : c[s] = c\}$. (Thus $[\![\cdot]\!]$ takes s to its fixed points on the update function $[\cdot]$.)

The preceding defines (i) a proposition map $\langle L, \mathcal{P}(C), [\![\cdot]\!]\rangle$ given an arbitrary commutative idempotent conversation system $\langle L, C, [\cdot]\rangle$, and (ii) a injective function f from $C \to \mathcal{P}(C)$. It remains to show that for all $c \in C$ and $s \in L$, $f(c[s]) = f(c) \cap [\![s]\!]$.

First we show that if $c_1 \in f(c[s])$, then $c_1 \in f(c) \cap [\![s]\!]$. Suppose $c_1 \in f(c[s])$. (i) Then $c_1 \in \{c' \in C : c[s] \leq c'\}$. So $c[s] \leq c_1$. By definition $c \leq c[s]$. So $c \leq c[s] \leq c_1$. Hence by transitivity $c \leq c_1$, hence $c_1 \in f(c)$. (ii) Now since $c[s] \leq c_1$, there exists some S such that $c[s][S] = c_1$. So $c[s][S][s] = c_1[s]$. By commutativity, $c[s][S][s] = c[S][s][s]$, which by idempotence equals $c[S][s]$, which by commutivity equals $c[s][S]$. So $c[s][S][s] = c[s][S]$. Here we substitute c_1 for $c[s][S]$, and we have $c_1[s] = c_1$. From this it follows that $c_1 \in [\![s]\!]$, since the latter just is $\{c \in C : c[s] = c\}$. So from (i) and (ii) we have $c_1 \in f(c) \cap [\![s]\!]$, the desired result.

Now let us show that if $c_1 \in f(c) \cap [\![s]\!]$, then $c_1 \in f(c[s])$. This is equivalent to showing that if $c_1[s] = c_1$ and $c \leq c_1$, then $c[s] \leq c_1$. Suppose $c \leq c_1$. Then there is some S such that $c[S] = c_1$. Suppose also $c_1[s] = c_1$. Then we have $c[S] = c_1[s] = c_1$. Therefore $c[S][s] = c_1$. By commutativity $c[s][S] = c_1$. And that means $c[s] \leq c_1$; and therefore $c_1 \in f(c[s])$.

□

The left-right direction completes the proof:

Fact 5.2 If a conversation system is static, then it is commutative and idempotent.

Proof. Any static system is idempotent and commutative, since intersection is idempotent and commutative. □

5 Commutativity

(1) a. Harry is married. Harry's spouse is a great cook.
b. ?Harry's spouse is a great cook. Harry is married.

(2) a. [A man]$_1$ walked in. He$_1$ was tall.
b. ?He$_1$ was tall. [A man]$_1$ walked in.

(3) a. Billy might be at the door.... Billy is not at the door.
b. ?Billy is not at the door... Billy might be at the door.

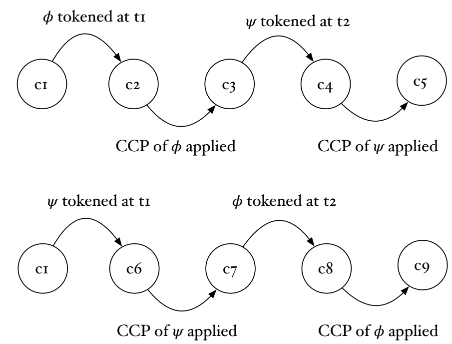

Figure 3: Merely reversing the order of sentences in natural language conversation does not result in commutation.

6 Information-sensitivity characterized

Def 10. An INFORMATION-RELATIVE PROPOSITION MAP is a quadruple $\langle L, C, P, [\![\cdot]\!]\rangle$, where L is a set of sentences, C is a set of contexts, P is a set of propositions, and $[\![\cdot]\!]$ is a mapping with $[\![\cdot]\!]$: $L \times C \to P$.

Def 11. A conversation system $\langle L, C, [\cdot]\rangle$ is INFORMATION-SENSITIVE if and only if there exists a set of sets P, an information-sensitive proposition map $\langle L, C, P, [\![\cdot]\!]\rangle$, and a one-to-one function f from C to P such that for all $c \in C$ and $s \in L$, $f(c) \cap [\![s]\!]^c = f(c[s])$.

Def 12. A conversation system $\langle L, C, [\cdot]\rangle$ is MONOTONIC just in case for all $s_i \in L$ and $c \in C$, if $c[s_i] \neq c$, then for all ordered sequences $s_1...s_n$ of elements of L, $c[s_i][s_1]...[s_n] \neq c$.[4]

Then the observation is this:

Fact 6. A conversation system is information-sensitive just in case it is monotonic.

Proof. Recall the definition of $\leq_U$ in the proof of Fact 5 above:

Def 8. For any conversation system U, and $c, c' \in C_U$, $c \leq_U c'$ iff there exist $s_1 \ldots s_n \in L_U$ such that $c[s_1]....[s_n] = c'$, or $c = c'$.

Observe that a conversation system $U = \langle L, C, [\cdot]\rangle$ is monotonic just in case $\leq_U$ is a partial order. Thus it suffices to show that U is information-sensitive iff $\leq_U$ is a partial order. We first show that if U is information-sensitive, the corresponding $\leq$ is a partial order. Reflexivity and transitivity are immediate consequences of the definition of $\leq$. For anti-symmetry simply note that for c, d in C' $c \leq d$ only if $c \supseteq d$. It follows that if $c \leq d$ and $d \leq c$, $c = d$.

We now show that if $\leq$ is a partial order, then U is information-sensitive. Let $f : C \to \mathcal{P}(C)$ be such that $f(c) : \{c' : c \leq c'\}$. Note that f is injective, as we showed in our proof of Fact 5 using only the fact that $\leq$ is a partial order. Now consider the information-relative proposition map $\langle L, C, \mathcal{P}(C), [\![\cdot]\!]\rangle$ where $[\![s]\!]$ is the minimal mapping such that for all $c \in C$ and $s \in L$, $[\![s]\!]^c = f(c[s])$. It remains to establish that for all $c \in C$ and $s \in L$, $f(c) \cap [\![s]\!]^c = f(c[S])$. First, we show that $[\![s]\!]^c \subseteq f(c)$. Note that $[\![s]\!]^c = f(c[s])$ and $c \leq c[s]$. Now, suppose $c' \in f(c[s])$, then, by definition, $c[s] \leq c'$. So, by transitivity of $\leq$, $c \leq c'$,and thus $c' \in f(c)$. So $[\![s]\!]^c \subseteq f(c)$. It follows immediately that: $f(c) \cap [\![s]\!]^c = [\![s]\!]^c = f(c[s])$.

□

Appendix

Fact 7. If a conversation system is incremental, then it is static.

Proof. Suppose $\langle L, C, [\cdot]\rangle$ is an incremental conversation system. Then there exists a proposition map $\langle L, P, [\![\cdot]\!]\rangle$ and a one-to-one function f from C to $\mathcal{P}(P)$ such that for all $c \in C$ and $s \in L$, $f(c) \cup \{[\![s]\!]\} = f(c[s])$. Consider the proposition map $\langle L, \mathcal{P}(P), [\![\cdot]\!]'\rangle$ such that $[\![s]\!]' = P \backslash \{[\![s]\!]\}$ $(= \{[\![s]\!]\}^c)$, and consider the function $f' : C \to \mathcal{P}(P)$ such that $f'(c) = f(c)^c$. Then for all $c \in C$ and $s \in L$:

$$f(c) \cup \{[\![s]\!]\} = f(c[s])$$
$$(f(c) \cup \{[\![s]\!]\})^c = f(c[s])^c$$
$$f(c)^c \cap \{[\![s]\!]\}^c = f(c[s])^c$$
$$f'(c) \cap [\![s]\!]' = f'(c[s])$$

□

Fact 8. Not every intersective system is incremental.

Proof. Consider an intersective conversation system $\langle L, C, [\cdot]\rangle$ such that:

(i) $c_1[p \wedge q][p] = c_1[p \wedge q]$

(ii) $c_1[p \wedge q] \neq c_1[p] \neq c_1$.

Suppose the system is incremental. Then there exists some proposition map $\langle L, P, [\![\cdot]\!]\rangle$ and a one-to-one function f from C to $\mathcal{P}(P)$ such that for all $c \in C$ and $s \in L$, $f(c) \cup \{[\![s]\!]\} = f(c[s])$. Given such a map, we know

$$f(c_1[p \wedge q][p]) = f(c_1) \cup \{[\![p \wedge q]\!]\} \cup \{[\![p]\!]\}$$

Since by (i) we have $c_1[p \wedge q][p] = c_1[p \wedge q]$, it follows that

$$f(c_1) \cup \{[\![p \wedge q]\!]\} \cup \{[\![p]\!]\} = f(c_1) \cup \{[\![p \wedge q]\!]\}$$

This entails that either $[\![p]\!] \in f(c_1)$ or $[\![p]\!] \in \{[\![p \wedge q]\!]\}$ (n.b., $\{[\![p]\!]\}$ is a singleton). Suppose the former. Then $f(c_1) \cup \{[\![p]\!]\} = f(c_1) = f(c_1[p])$; but since f is one-one, this result is incompatible with (ii), which says $c_1[p] \neq c_1$. So suppose instead $[\![p]\!] \in \{[\![p \wedge q]\!]\}$. But this entails $[\![p]\!] = [\![p \wedge q]\!]$, meaning $f(c_1[p \wedge q]) = f(c_1[p])$. Since f is one-one, this result is incompatible with (ii), which says $c_1[p \wedge q] \neq c_1[p]$.

□

[4] Don't confuse this notion with the notion of monotony used to define the Veltman static systems in §5 above. The latter applies only in the context of information lattices.