Berkeley, February 8, 2019

Computation with Atoms

Szymon Toruńczyk, University of Warsaw

includes work with Bojańczyk, Klin, Kopczyński, Lasota, Ochremiak,...

Atoms

a fixed *underlying* logical structure

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a fixed *underlying* logical structure

Examples:

$(\mathrm{N}, =)$ – *pure set*

$(\mathrm{Q}, \leq)$ – *dense order*

$(\mathrm{R}, +, \times, 0, 1)$ – *field of reals*

$(\mathrm{N}, +, \leq)$ – *Presburger arithmetic*

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Hereditarily definable set

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Examples

5 if $5 \in \text{Atoms}$

$\{a : a \in \text{Atoms}\}$

$\{a : a \in \text{Atoms},\ a \neq 7 \wedge a \neq 5\}$ if $5,7 \in \text{Atoms}$

$\{\{a : a \in \text{Atoms},\ a \neq b\} : b \in \text{Atoms}\}$

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| { hdef : variable,...,variable ∈ **Atoms,** first order formula }

in language of Atoms,
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$\{x,y\} \overset{\text{def}}{=} \{x\} \cup \{y\}$ $(x,y) \overset{\text{def}}{=} \{x, \{x,y\}\}$

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Hereditarily definable sets have finite descriptions

e.g. $\{\{a : a \in \text{Atoms}, a \neq b\} : b \in \text{Atoms}\}$

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equality of sets is decidable ⇔ the theory of Atoms is decidable

Hereditarily definable *X*

graphs *automata* *Turing machines*

graphs

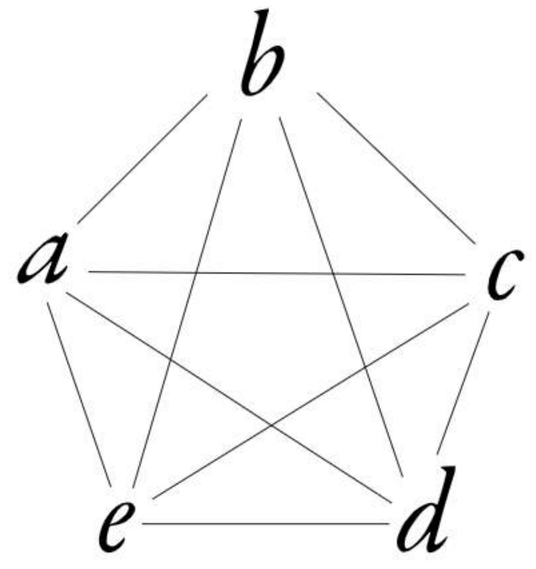
graphs

A pair (V,E) of hereditarily definable sets with E$\subseteq \binom{V}{2}$

infinite clique

vertices: $\{ a : a \in \text{Atoms} \}$

edges: $\{\{a,b\} : a,b \in \text{Atoms},\ a \neq b\}$



Johnson graph

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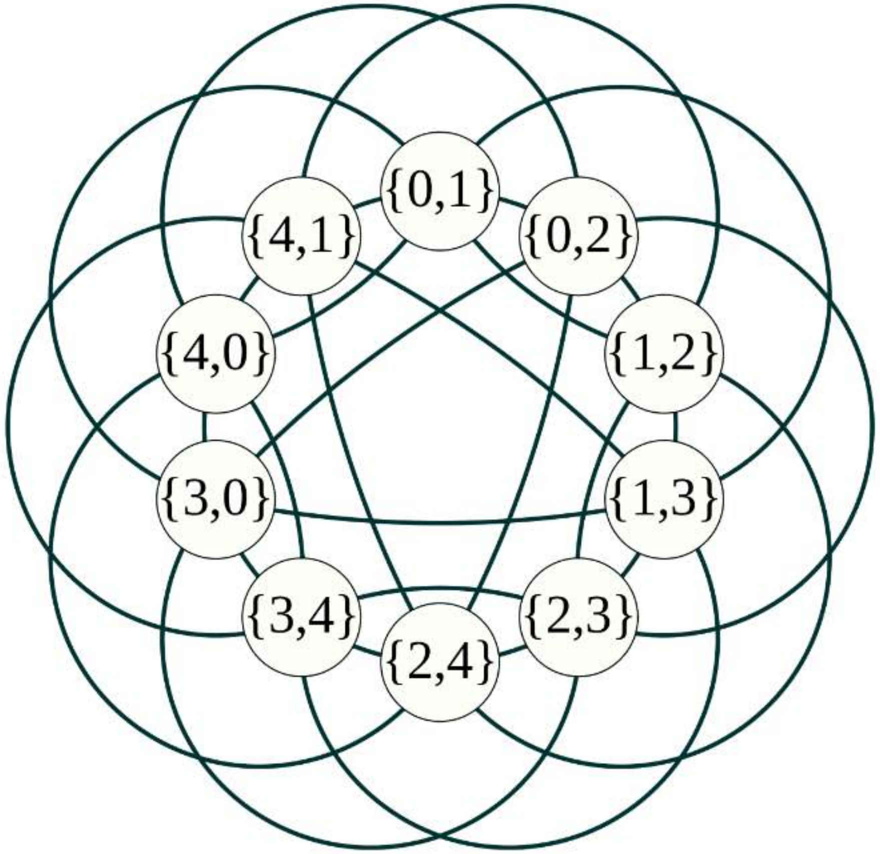

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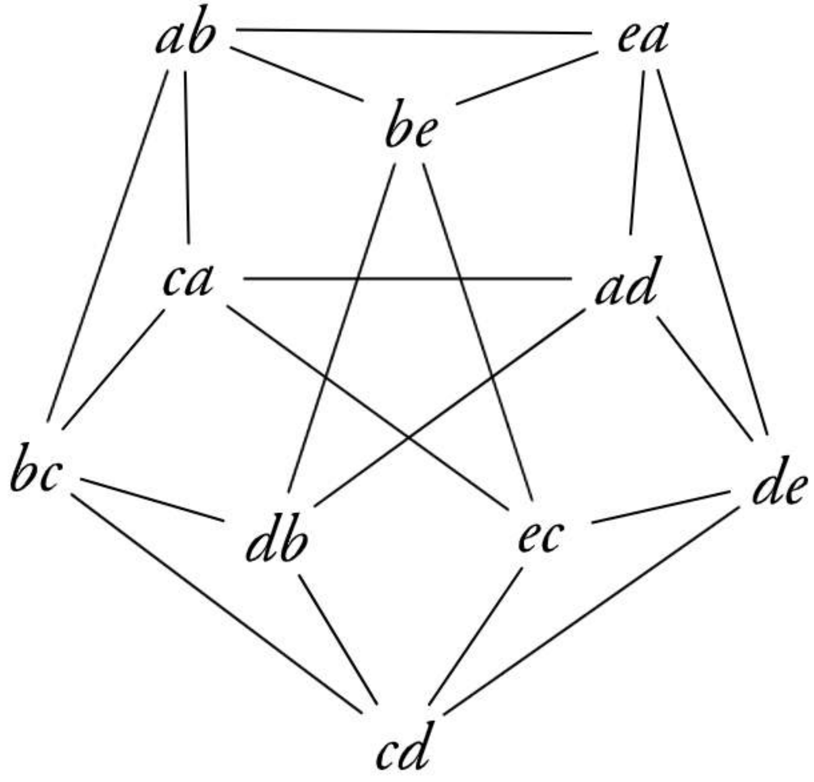

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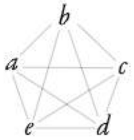

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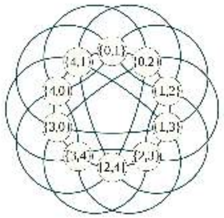

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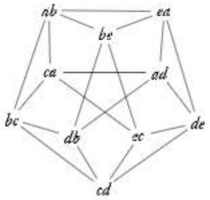

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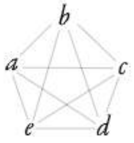

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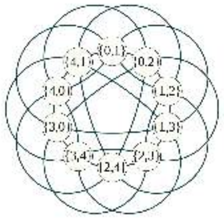

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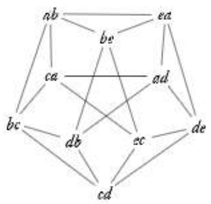

decision problems:

- connectedness
- 3-colorability
- homomorphism
- isomorphism
- ...

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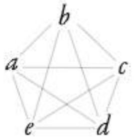

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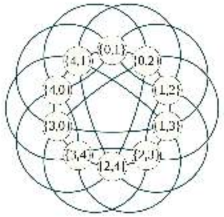

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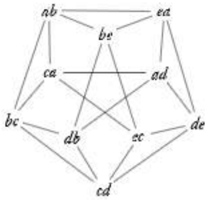

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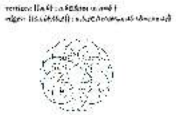

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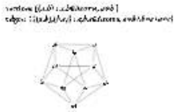

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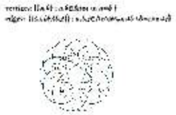
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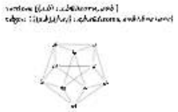
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A tuple of hereditarily definable sets

$$(\text{States}, \text{Alphabet}, \text{Initial}, \text{Accepting}, \delta)$$

where $\text{Initial}, \text{Accepting} \subseteq \text{States}$ and $\delta \subseteq \text{States} \times \text{Alphabet} \times \text{States}$

Atoms are $(\mathbf{Q},\leq)$.

An automaton can accept sequences

$$q_1\ q_2\ q_3\ q_4\ \dots\ q_n$$

such that $q_1<q_2<q_3<q_4<\dots<q_n$

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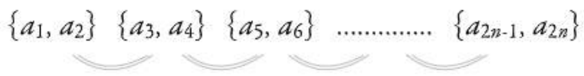

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computational problems:

- emptiness
- language equality
- minimization
- …

Turing machines

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when Atoms = $(\mathbf{Q}, \leq)$:

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when Atoms $=((\mathbf{Z}/2\mathbf{Z})^{\omega},+)$:

$$\mathrm{P} \neq \mathrm{NP}$$

separating language:

sequences of vectors $v_1\ v_2\ v_3\ v_4\ \ \ldots\ldots\ \ v_n$

which are linearly dependent

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- Cai-Furer-Immermann graphs
- universal algebra
- A is is not homogeneous in a finite relational/functional language

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History

1. In set theory, Fraenkel and Mostowski studied sets constructed on top of an underlying set of *urelementa* or *atoms*.

   "Hereditarily definable sets" are a special case of these, and have finite syntax.

2. Gabbay and Pitts (2002) rediscovered Fraenkel-Mostowski sets in the case of atoms ($\mathbf{N}$,=), in the context of *name binding in semantics,* and called them *nominal sets.*

3. Bojańczyk et al. (2011) rediscovered these sets in the case of *homogeneous atoms,* in the context of *automata theory* and called them *orbit-finite sets with atoms.*

4. Up to isomorphism, a structure is *hereditarily definable* ⇔ it *interprets* in Atoms

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$\{x, y\} \overset{\text{def}}{=} \{x\} \cup \{y\}$ $\quad (x, y) \overset{\text{def}}{=} \{x, \{x, y\}\}$

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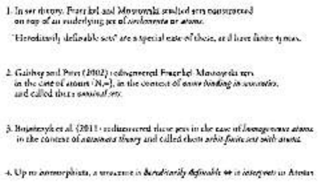

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| { hdef : variable,...,variable ∈ **Atoms**, first order formula } (first order formula in language of Atoms, with parameters)

| hdef ∪ hdef

Hereditarily definable sets have finite descriptions

e.g. $\{\{a : a \in \text{Atoms}, a \neq b\} : b \in \text{Atoms}\}$

→ can be input and processed by algorithms

equality of sets is decidable ⇔ the theory of Atoms is decidable

Hereditarily definable *X*

graphs *automata* *Turing machines*

(States, Alphabet, Initial, Accepting, δ)

(States, Alphabet, δ)

History

1. In set theory, Fraenkel and Mostowski studied sets constructed on top of an underlying set of *urelements* or *atoms*.

   "Hereditarily definable sets" are a special case of these, and have finite names.

2. Gabbay and Pitts (2002) rediscovered Fraenkel-Mostowski sets in the case of atoms (N,=), in the context of *name binding in semantics*, and called them *nominal sets*.

3. Bojańczyk et al. (2011) rediscovered these sets in the case of *homogeneous atoms* in the context of *automata theory* and called them *orbit-finite sets with atoms*.

4. Up to isomorphism, a structure is *hereditarily definable* ⇔ it *interprets* in Atoms

Computation with Atoms

Szymon Toruńczyk, University of Warsaw

includes work with Bojańczyk, Klin, Kopczyński, Lasota, Ochremiak,...

Atoms

a fixed *underlying* logical structure

Examples:

(N , =) – *pure set* ← main example in this talk

(Q , ≤) – *dense order*

(R , +, ×, 0, 1) – *field of reals*

(N , +, ≤) – *Presburger arithmetic*

Hereditarily definable set

Examples

5 if $5 \in$ Atoms

$\{a: a \in \text{Atoms}\}$

$\{a: a \in \text{Atoms}, a \neq 7 \wedge a \neq 5\}$ if $5,7 \in$ Atoms

$\{\{a: a \in \text{Atoms}, a \neq b\}: b \in \text{Atoms}\}$

$\{\{b: b \in \text{Atoms}, a<b \wedge b<c\}: a,c \in \text{Atoms}, a<c\}$ if Atoms = $(\mathbf{Q}, \leq)$

$\{x,y\} \overset{\text{def}}{=} \{x\} \cup \{y\}$ $(x,y) \overset{\text{def}}{=} \{x, \{x,y\}\}$

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Computational Problems

definable sets can be presented as input to algorithms

Computational Problems

definable sets can be presented as input to algorithms

- Graph reachability
- Deterministic automata minimisation
- Context-free grammar emptiness
- Tree/pushdown automata emptiness
- Graph planarity
- Graph isomorphism
- Graph 3–colorability
- Solvability of systems of equations over finite field
- Satisfiability of sets of clauses
- Constraint Satisfaction Problems over finite template
- Homomorphism problem

Reachability

Input: a hereditarily definable graph $G=(V,E)$, vertices $s,t \in V$

Decide: is t reachable from s?

decidable when atoms are $(\mathbf{N},=)$ or $(\mathbf{Q},\leq)$

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decidable when atoms are $(\mathbf{N},=)$ or $(\mathbf{Q},\leq)$

undecidable for Atoms = $(\mathbf{N},+1)$ (~Minsky machines)

Algorithm pseudocode

```
function reachability(V,E,s,t)

R_0 := {s};
n=0;
repeat
    $R_{n+1}$ := $R_n$ ∪ {w | (v,w)∈E, v∈$R_n$, w∈V};
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until ($R_n$=$R_{n-1}$);
return (t∈$R_n$)
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Terminates for *ω-categorical* Atoms

tes for ω-*categorical* A

Examples: (**N**,=),(**Q**,≤), Rado graph

Theorem [Ryll-Nardzewski, Engeler, Svenonius]

a structure Atoms is ω–*categorical* if and only if for all $n \in \mathbf{N}$

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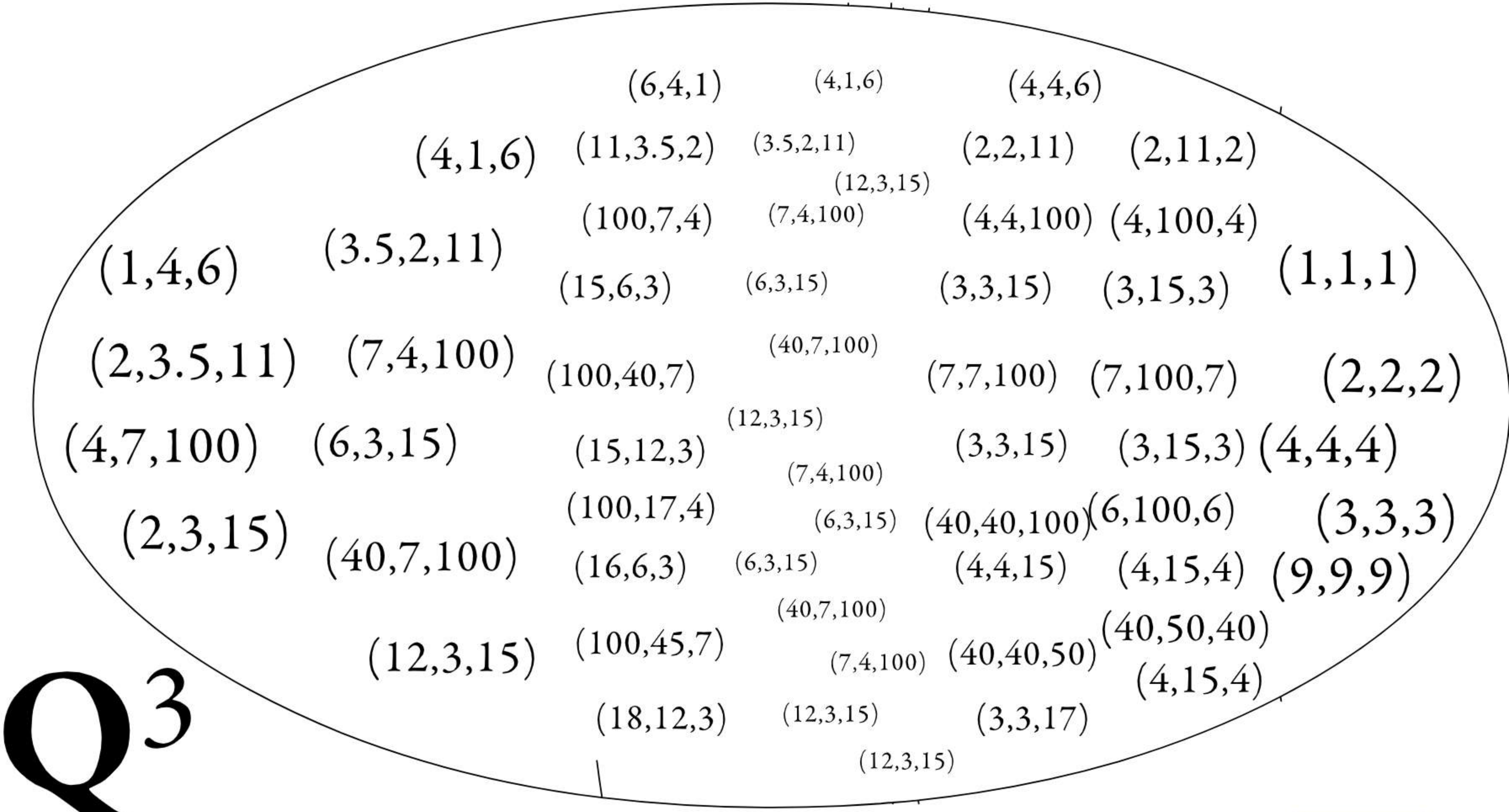

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(12,3,15)

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(1,4,6) (3.5,2,11) (1,1,1)

(15,6,3) (6,3,15) (3,3,15) (3,15,3)

(40,7,100)

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(12,3,15)

$\mathbf{Q}^3$

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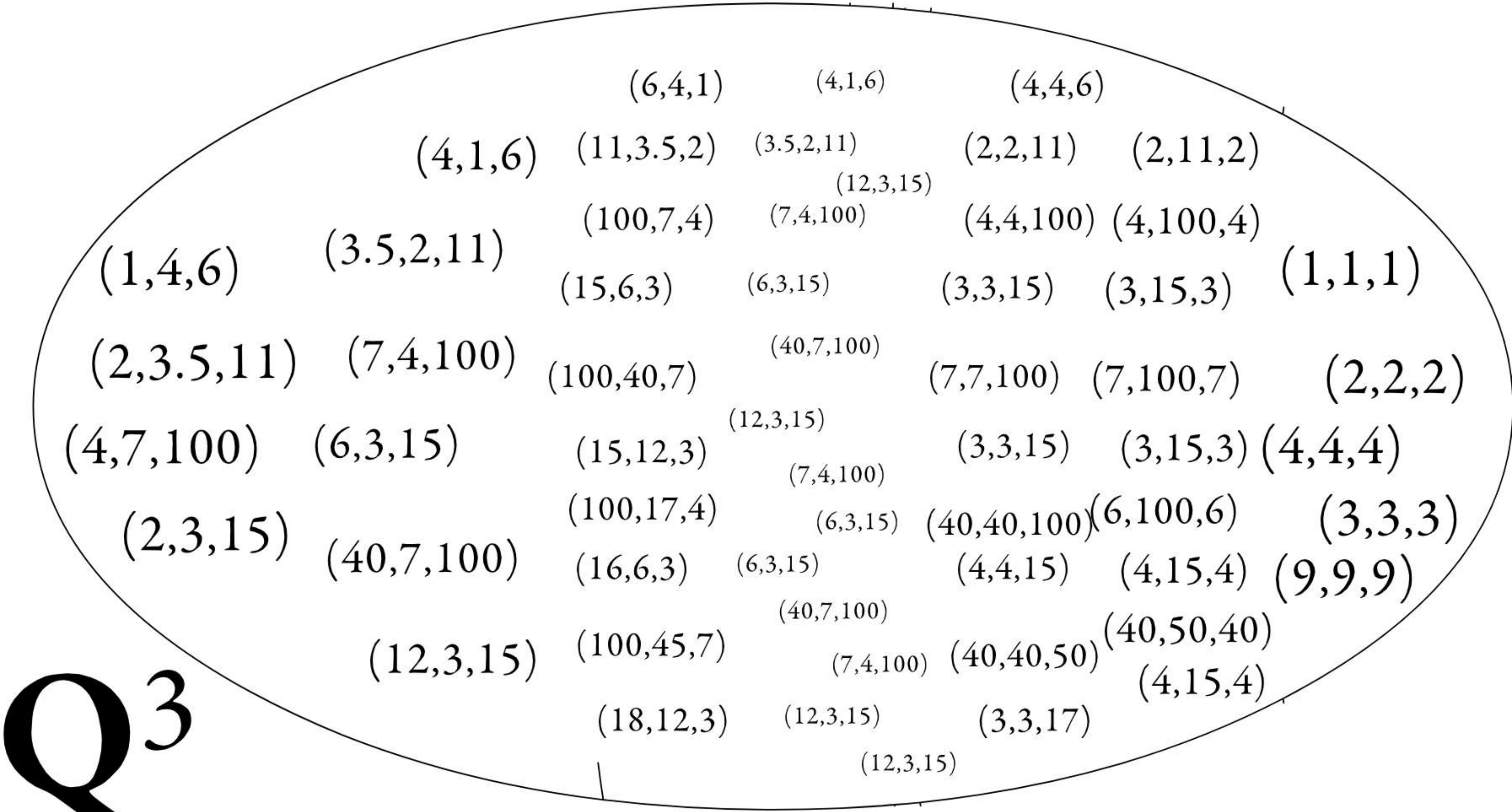

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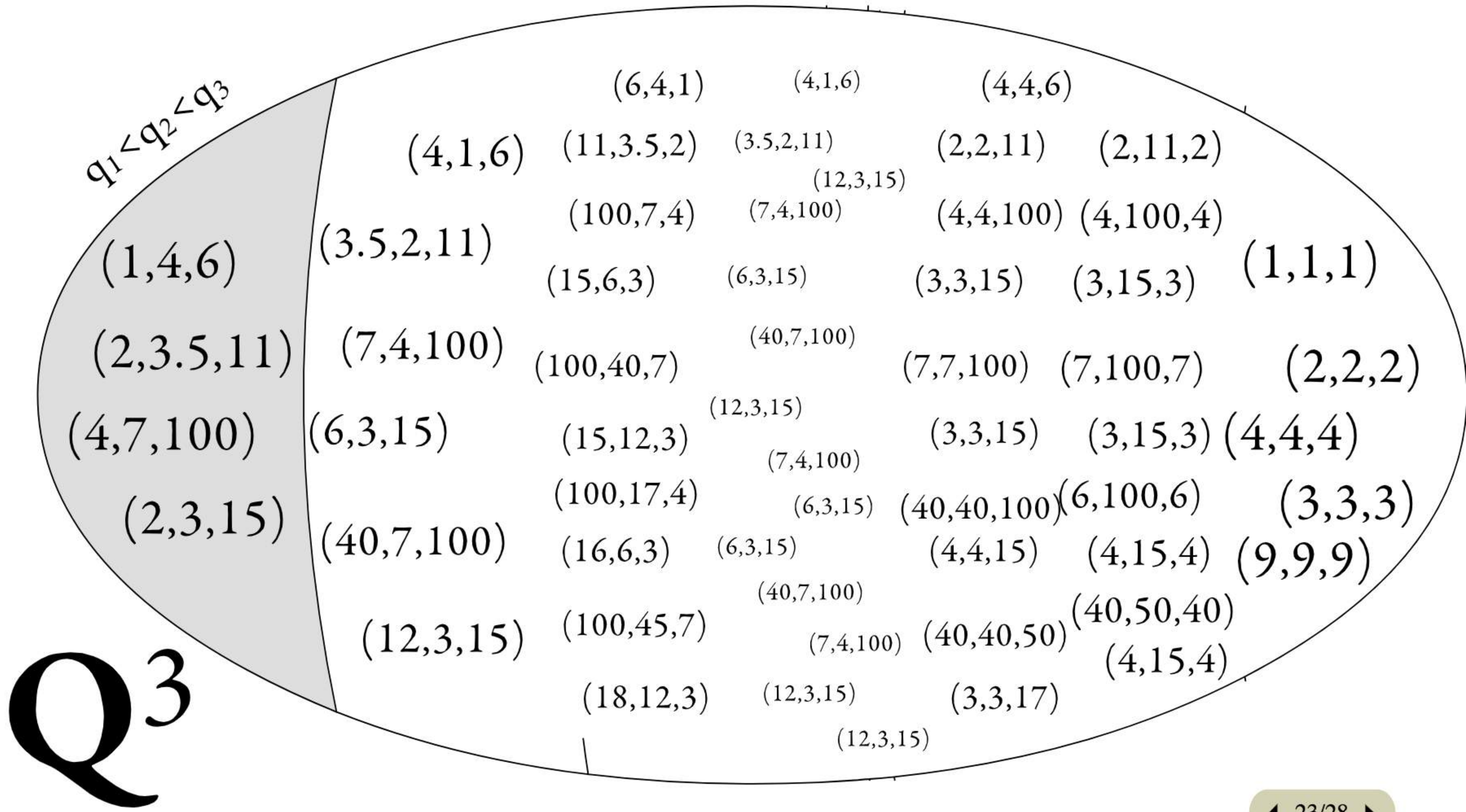

$q_1<q_2<q_3$

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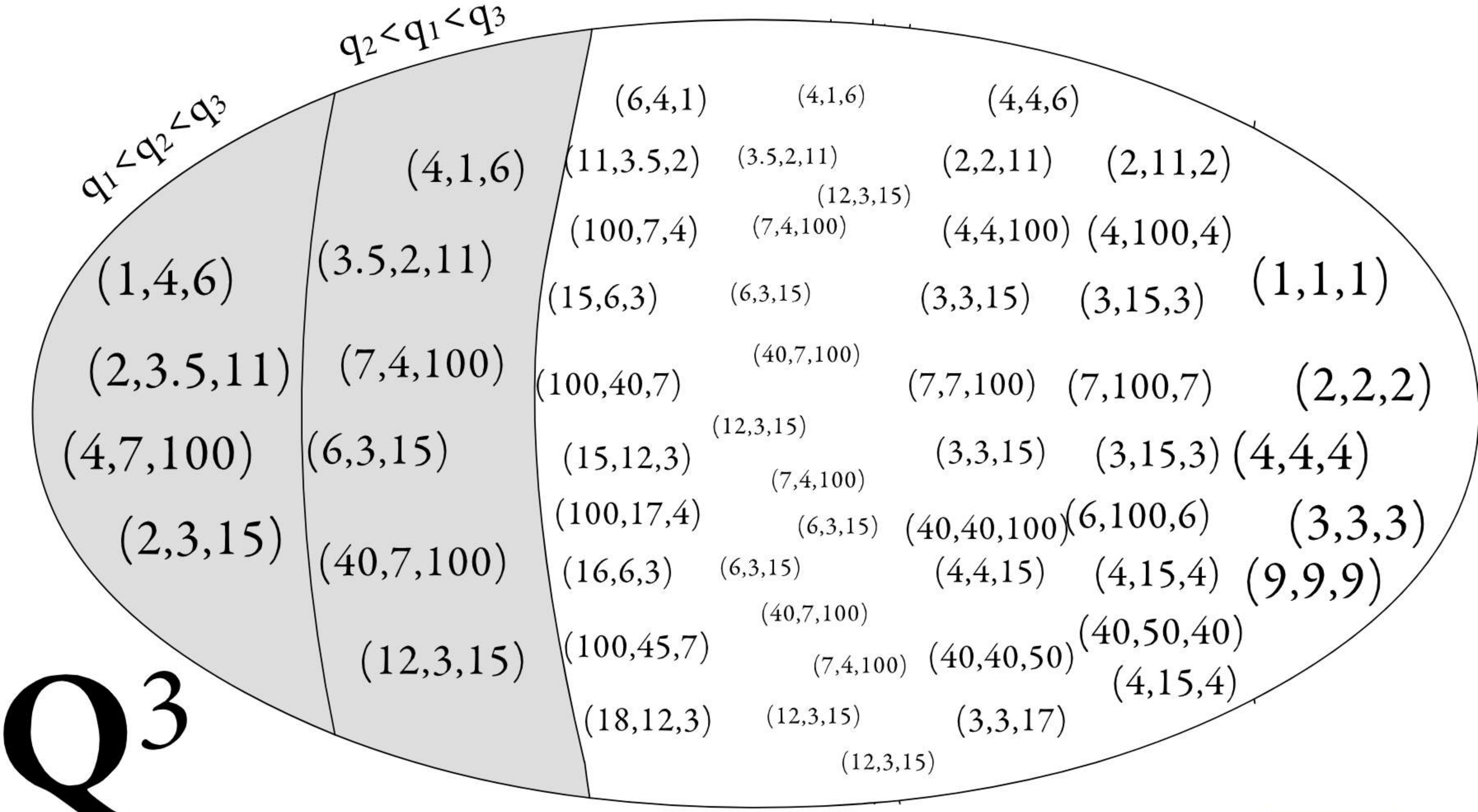

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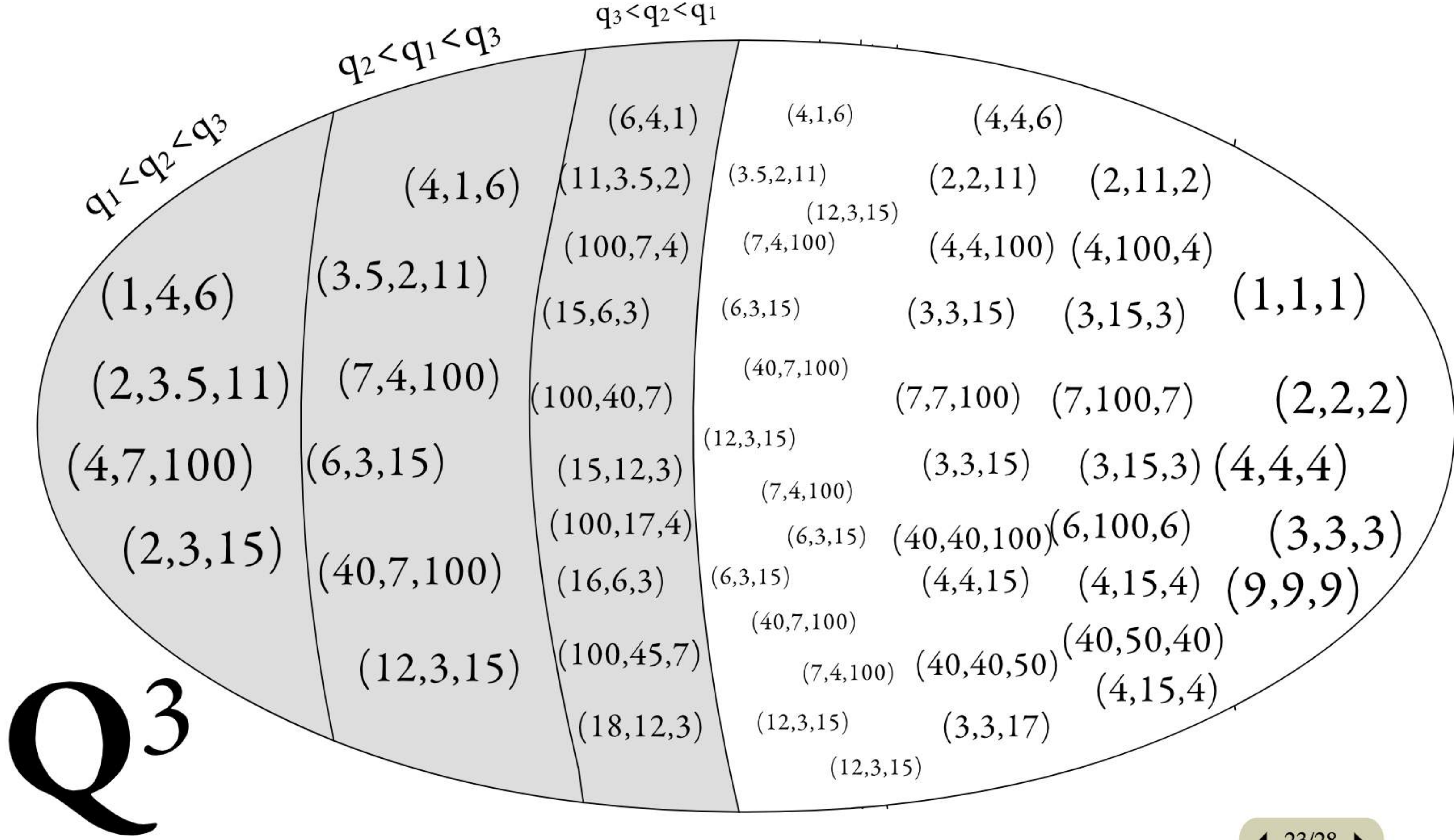
$\mathbf{Q}^3$

nitely many orbits under the action of

Atoms = ($\mathbf{Q}$,≤)



$q_3 < q_2 < q_1$

$q_2 < q_1 < q_3$

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(1,4,6)
(2,3.5,11)
(4,7,100)
(2,3,15)

(4,1,6)
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(6,3,15) (3,3,15) (3,15,3)
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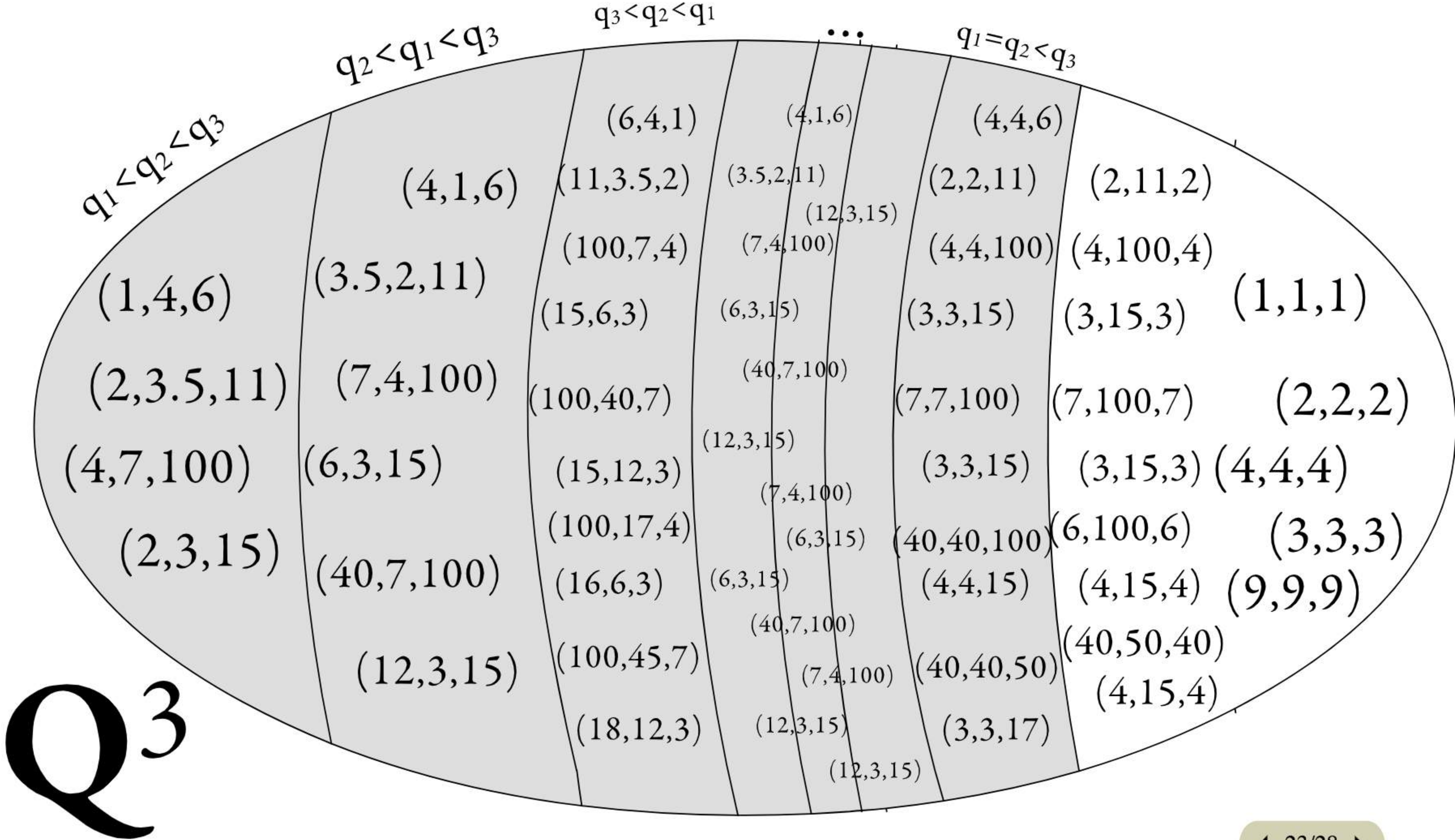
Atoms = $(\mathbf{Q},\leq)$



$q_1<q_2<q_3$

(1,4,6)
(2,3.5,11)
(4,7,100)
(2,3,15)

$q_2<q_1<q_3$

(4,1,6)
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…

(4,1,6)
(3.5,2,11)
(12,3,15)
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(7,4,100)
(6,3,15)
(6,3,15)
(40,7,100)
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$q_1=q_2<q_3$

(4,4,6)
(2,2,11)
(4,4,100)
(3,3,15)
(7,7,100)
(3,3,15)
(40,40,100)
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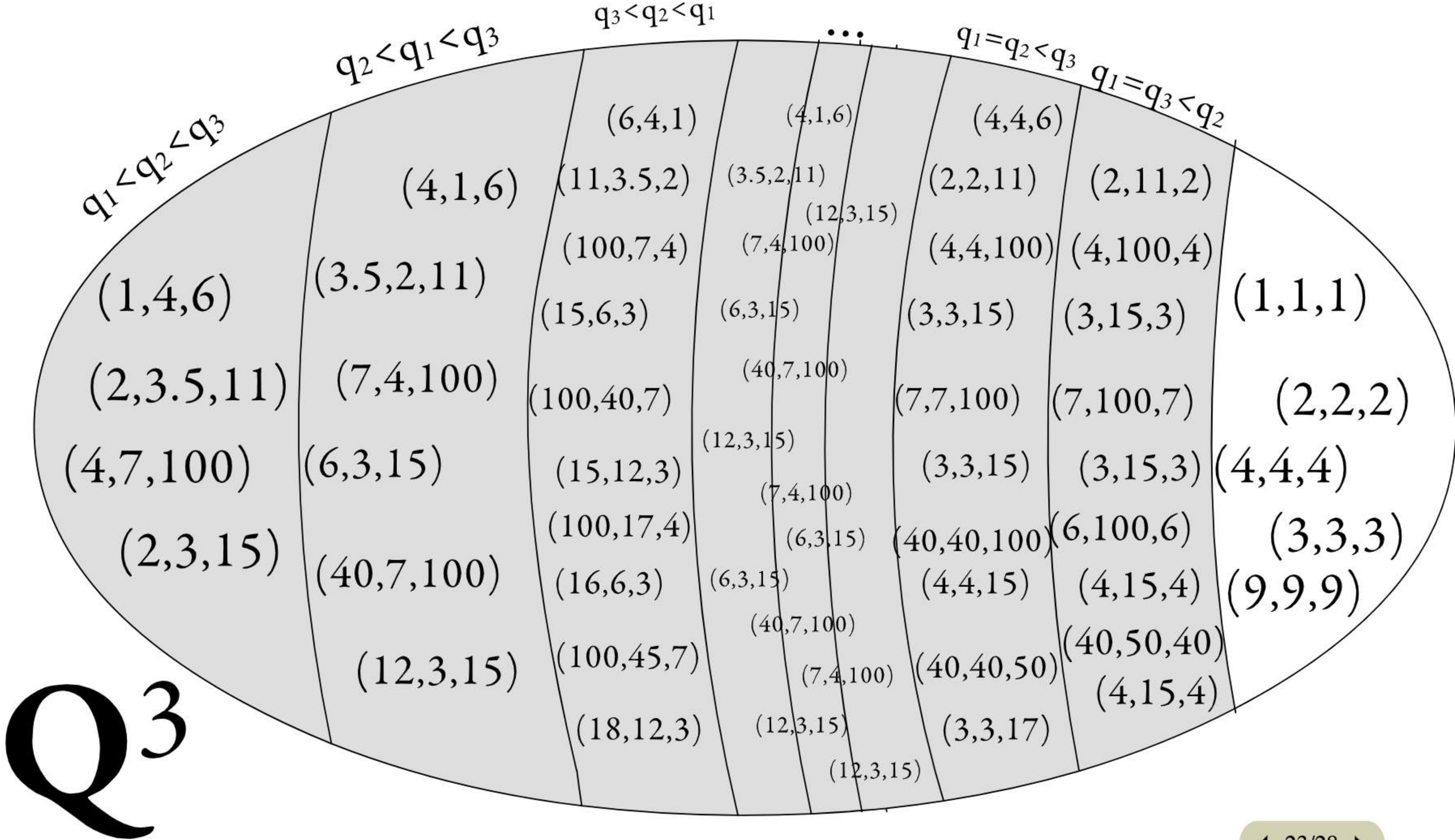
$\mathbf{Q}^3$

nitely many orbits under the action of

$\text{Atoms} = (\mathbf{Q}, \leq)$



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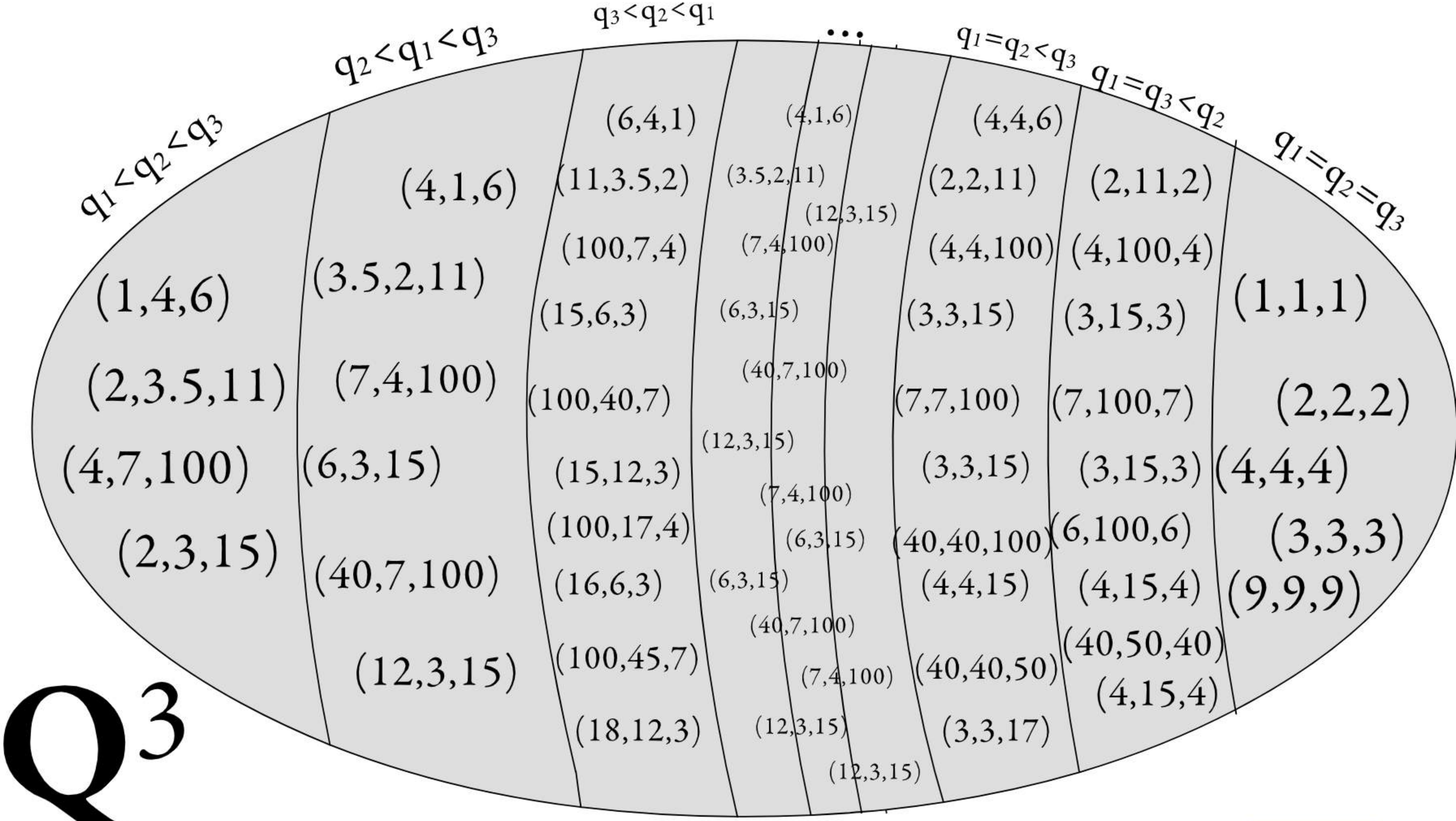
Atoms = (**Q**,≤)



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q₃<q₂<q₁: (6,4,1) (11,3.5,2) (100,7,4) (15,6,3) (100,40,7) (15,12,3) (100,17,4) (16,6,3) (100,45,7) (18,12,3)

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q₁=q₂<q₃: (4,4,6) (2,2,11) (4,4,100) (3,3,15) (7,7,100) (3,3,15) (40,40,100) (4,4,15) (40,40,50) (3,3,17)

q₁=q₃<q₂: (2,11,2) (4,100,4) (3,15,3) (7,100,7) (3,15,3) (6,100,6) (4,15,4) (40,50,40) (4,15,4)

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Examples: (**N**,=),(**Q**,≤), Rado graph

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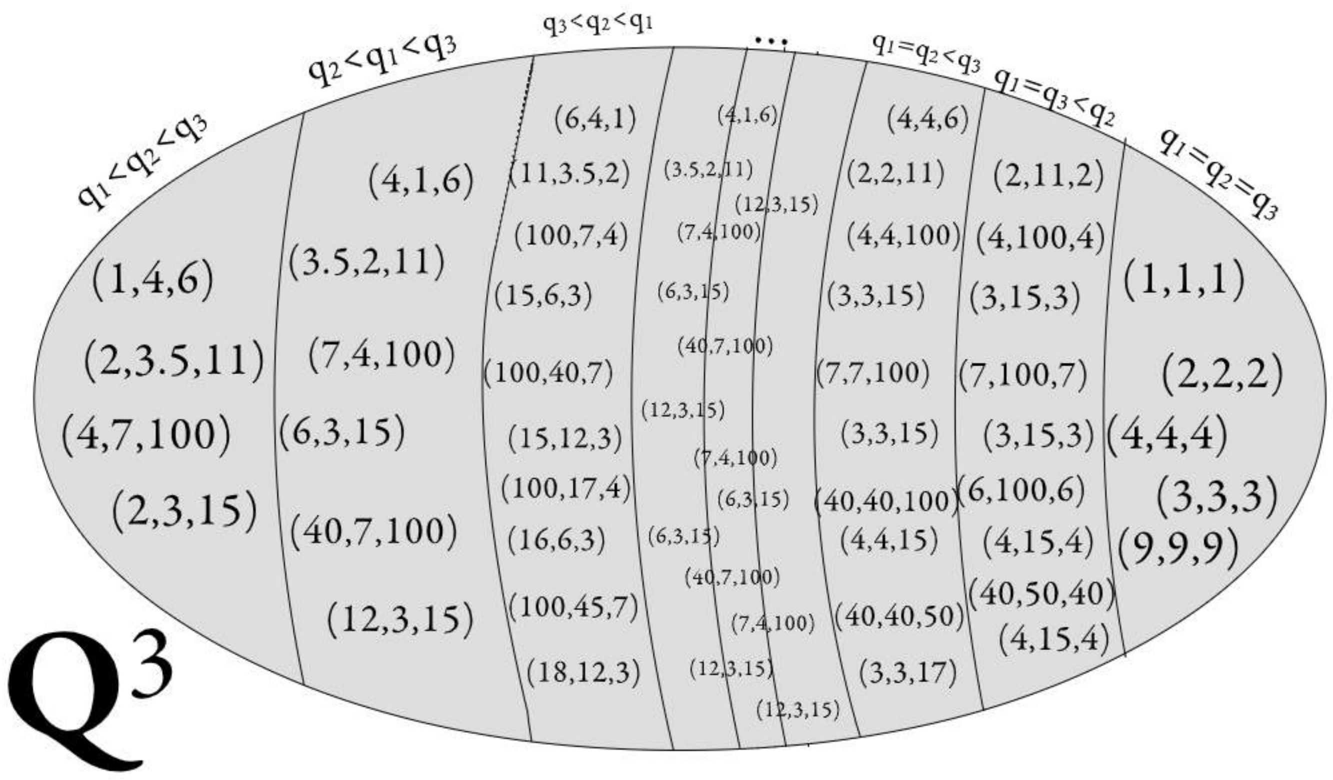

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minates for *ω-categorical* Atoms: $R_0 \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \ldots \subseteq V$ are invariant subsets of V, which has finitely many orbits.

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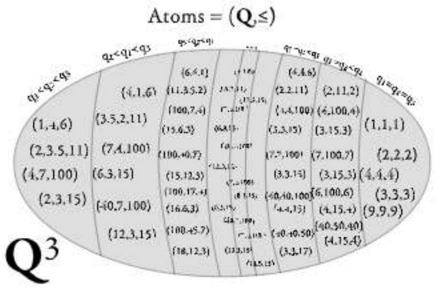

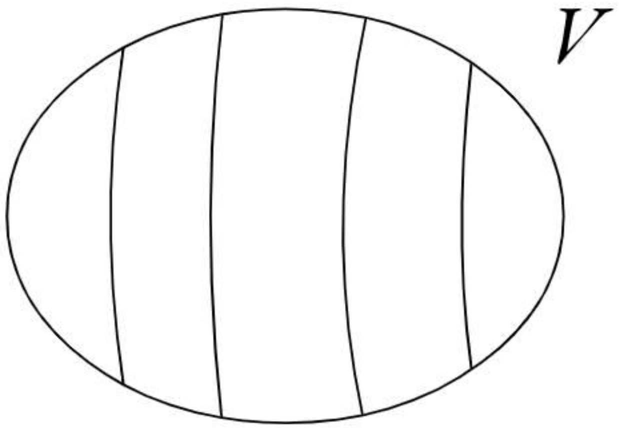

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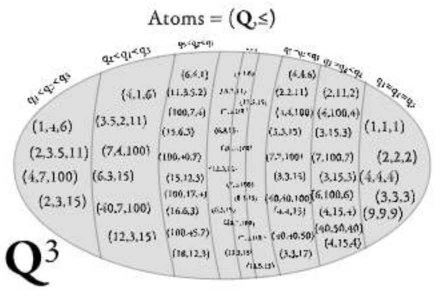

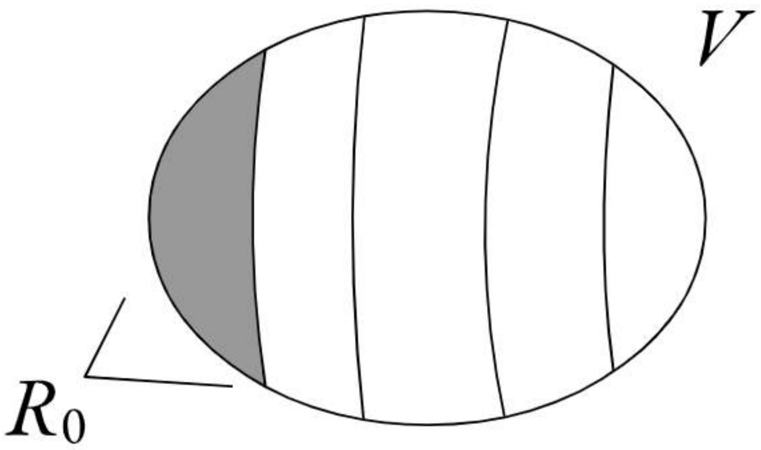

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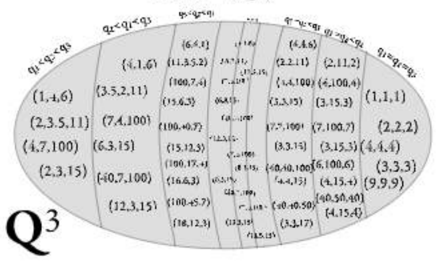
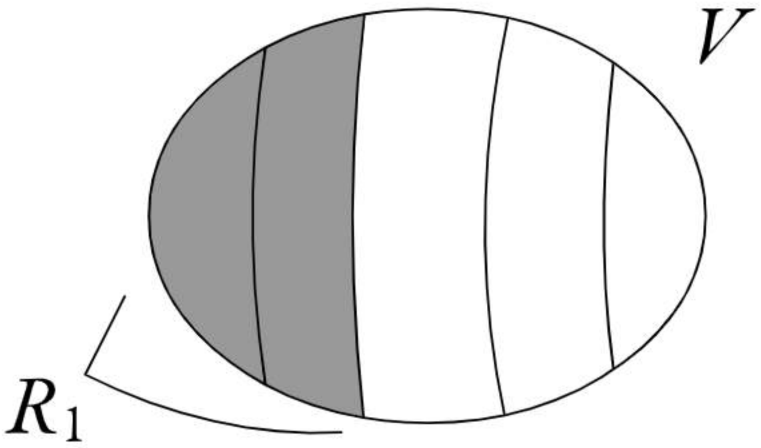
Atoms = (**Q**,≤)





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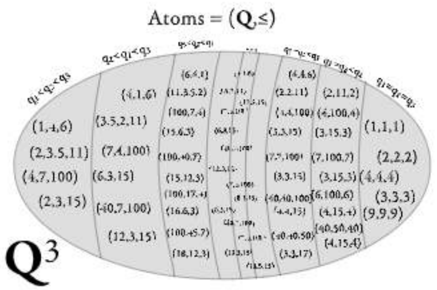
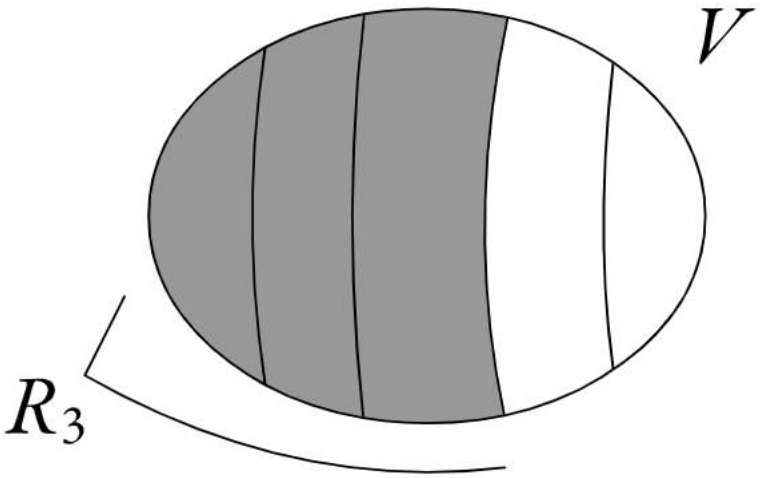
Examples: (**N**,=),(**Q**,≤), Rado graph

Theorem [Ryll-Nardzewski, Engeler, Svenonius]
a structure Atoms is *ω–categorical* if and only if for all $n \in \mathbf{N}$

Atoms^n *has finitely many orbits under the action of Aut(*Atoms*)*





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function reachability(V,E,s,t)

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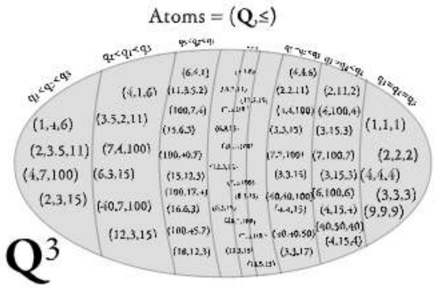

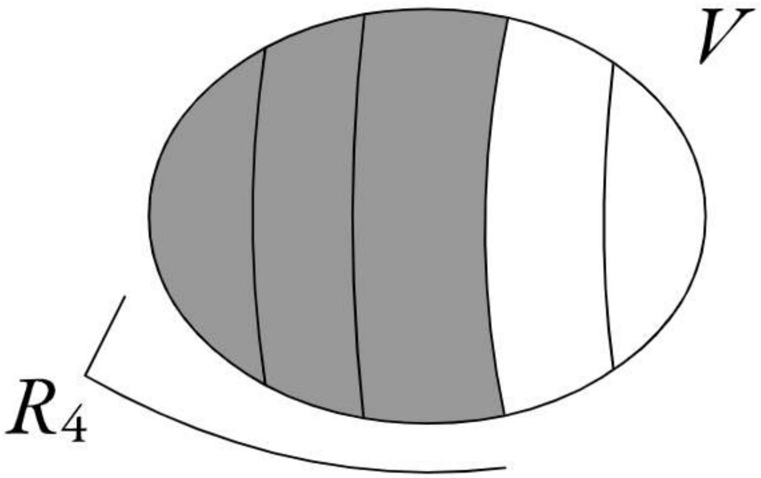

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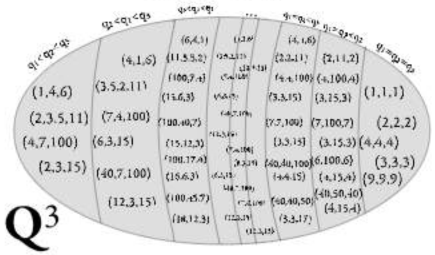
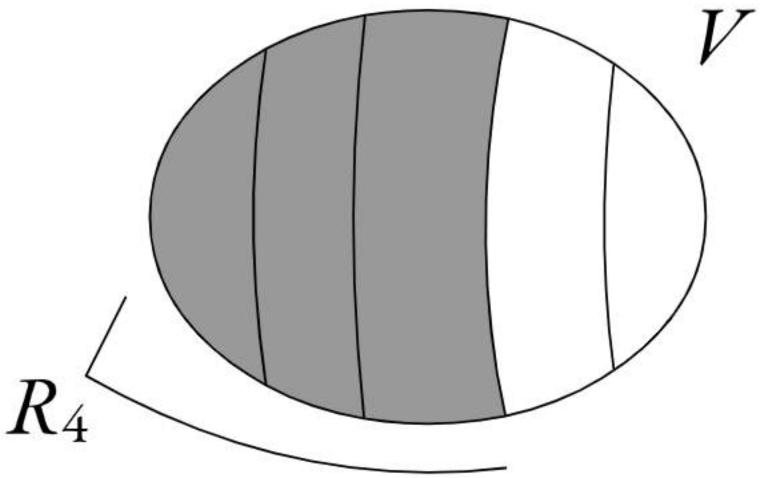
Atoms = ($\mathbf{Q}$,≤)





Reachability

Input: a hereditarily definable graph $G=(V,E)$, vertices $s,t \in V$
Decide: is t reachable from s?

decidable when atoms are $(\mathbf{N},=)$ or $(\mathbf{Q},\leq)$

undecidable for Atoms = $(\mathbf{N},+1)$ (~Minsky machines)

Algorithm pseudocode

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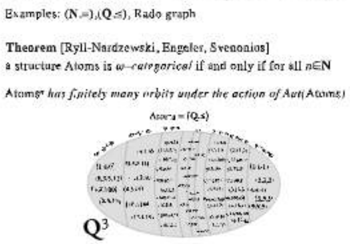

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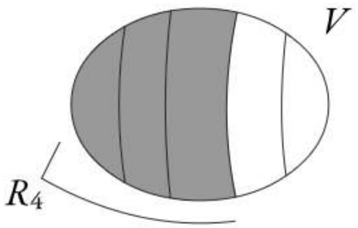

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   $R_{n+1}$ := $R_n \cup \{w \mid (v,w) \in E,\ v \in R_n,\ w \in V\}$;
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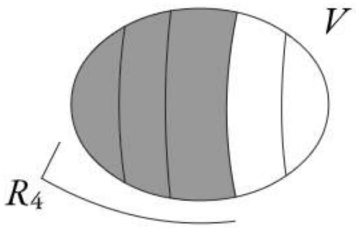

Computational Problems

definable sets can be presented as input to algorithms

- Graph reachability

Reachability

- Deterministic automata minimisation
- Context-free grammar emptiness
- Tree/pushdown automata emptiness
- Graph planarity
- Graph isomorphism
- Graph 3–colorability
- Solvability of systems of equations over finite field
- Satisfiability of sets of clauses
- Constraint Satisfaction Problems over finite template
- Homomorphism problem

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Input: A graph G definable over Atoms = $(\mathbf{N},=)$

Decide: Is G 3-colorable?

Is this problem decidable?

Example

Is the following graph 3-colorable?

$$V = \{\ (a,b) : a,b \in \text{Atoms},\ a \neq b\ \}$$

$$E = \{\ \{(a,b),(b,c)\} : a,b,c \in \text{Atoms},\ a \neq b \wedge b \neq c \wedge a \neq c\ \}$$

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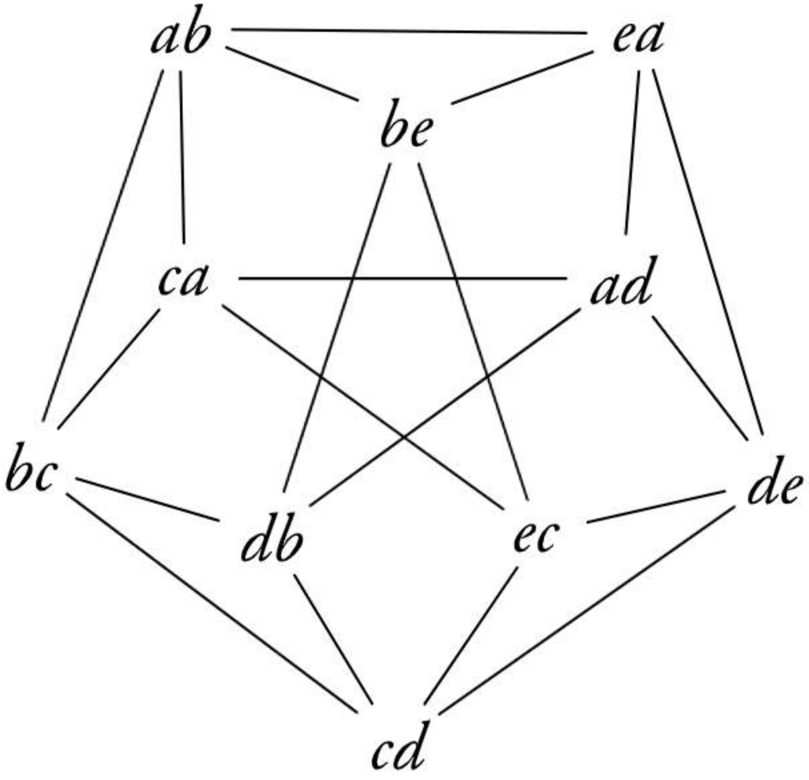

Input: A graph *G* definable over Atoms = ($\mathbf{N}$,=)

Decide: Is *G* 3-colorable?

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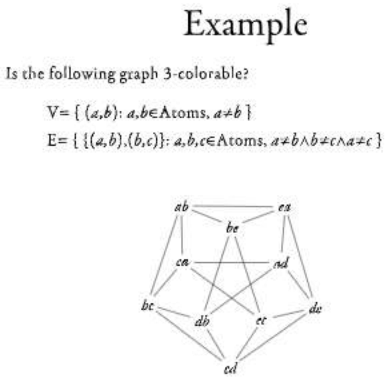

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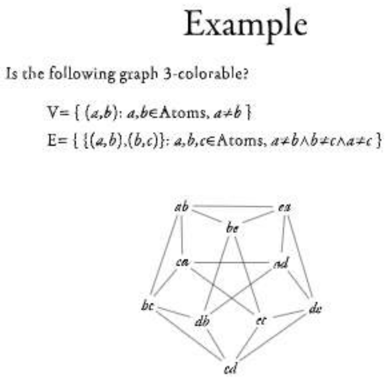

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Proof of decidability uses Ramsey theory [Bodirsky, Pinsker, Tsankov'13]
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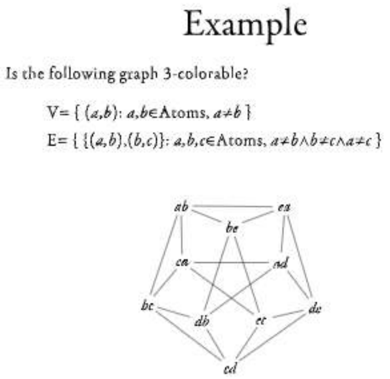

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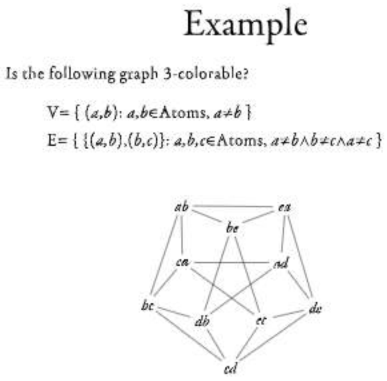

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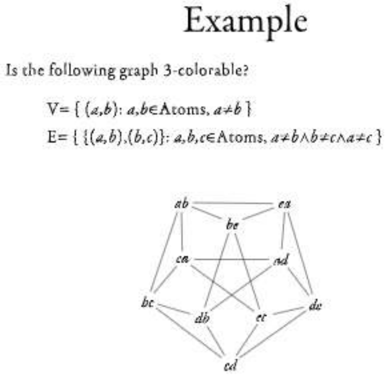
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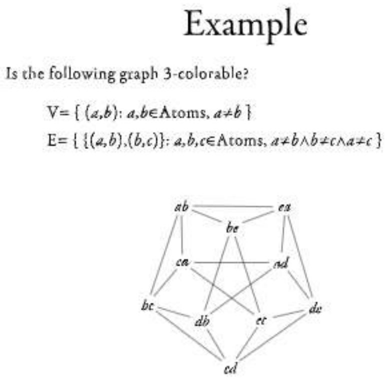

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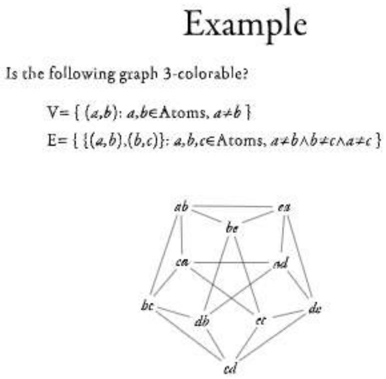

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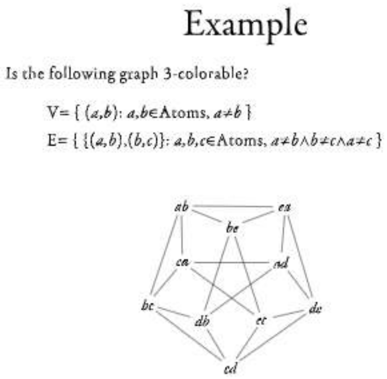

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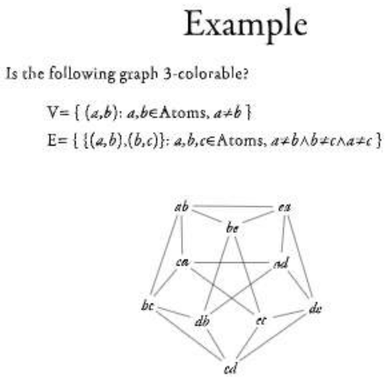

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closed
invariant

compact
$\mathrm{Aut}(\mathbf{Q},\leq)$ action

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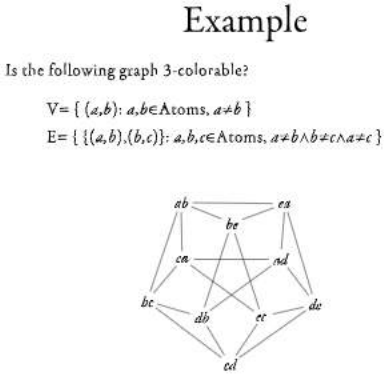

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By Pestov:

if there is a 3-coloring of G then there is an invariant one.

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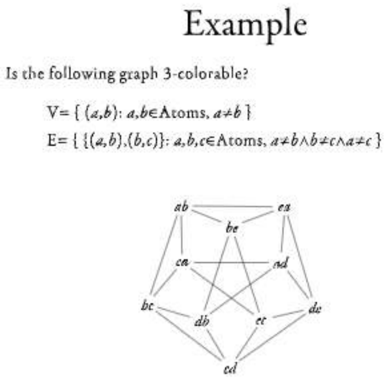

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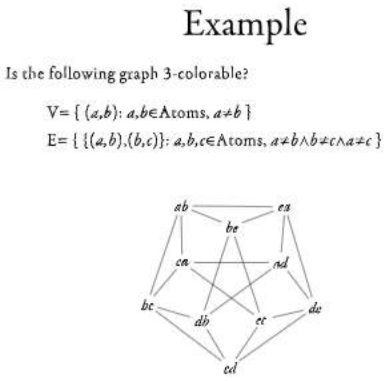
if there is a 3-coloring of G then there is an invariant one.

Check 3-colorolability of the finite graph G/Aut$(\mathbf{Q},\leq)$ of orbits of G. ■

Input: A graph G definable over Atoms = $(\mathbf{N},=)$

Decide: Is G 3-colorable?

Is this problem decidable?



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Also works when Atoms are a Ramsey structure (KPT theorem).

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- ✓ Graph reachability

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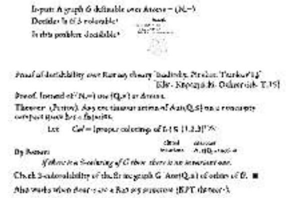

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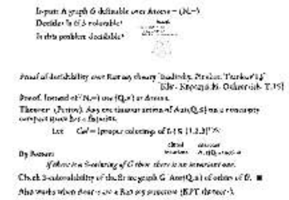
(ω-categoricity)

- ✓ Graph reachability

- ✓ Deterministic automata minimisation
- ✓ Context-free grammar emptiness
- ✓ Tree/pushdown automata emptiness
- Graph planarity
- Graph isomorphism
- Graph 3–colorability



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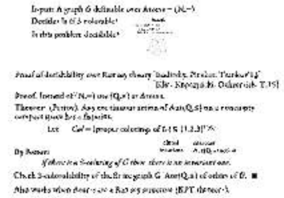

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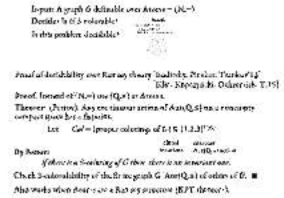

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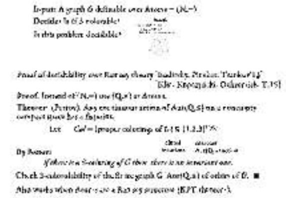

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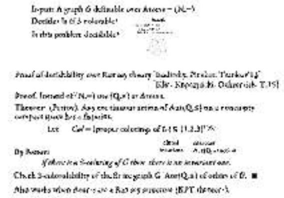

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X = ∅;

for a in Reals
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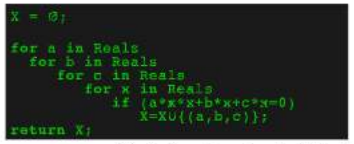

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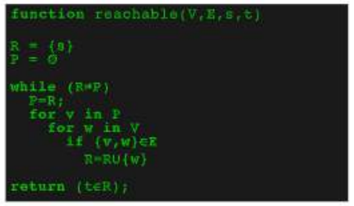

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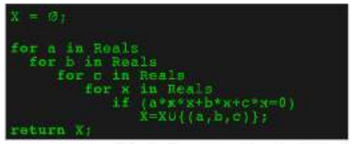

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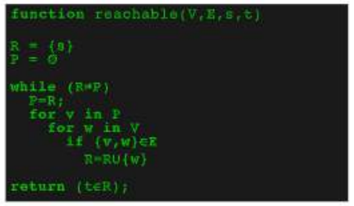

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Berkeley, February 8, 2019

Computation with Atoms

Szymon Toruńczyk, University of Warsaw

includes work with Bojańczyk, Klin, Kopczyński, Lasota, Ochremiak,...

Atoms

a fixed *underlying* logical structure

Examples:

$(\mathbb{N}, =)$ – *pure set* ← main example in this talk

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Hereditarily definable set

Examples

5 if $5 \in$ Atoms

$\{a : a \in \text{Atoms}\}$

$\{a : a \in \text{Atoms}, a \neq 7 \wedge a \neq 5\}$ if $5,7 \in$ Atoms

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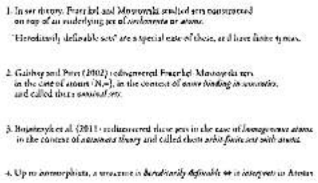
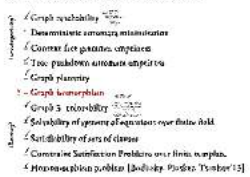


Computational Problems



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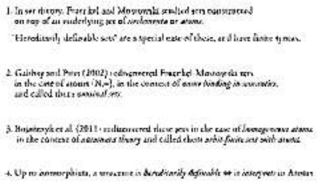

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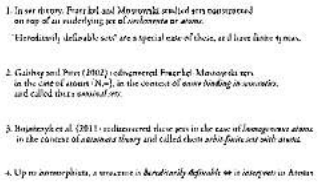

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(States, Alphabet, Initial, Accepting, δ)

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History

1. In set theory, Fraenkel and Mostowski studied sets constructed on top of an underlying set of *urelements* or *atoms*.

   "Hereditarily definable sets" are a special case of these, and have finite types.

2. Gabbay and Pitts (2002) rediscovered Fraenkel-Mostowski sets in the case of atoms (N,=), in the context of *name binding in semantics*, and called them *nominal sets*.

3. Bojańczyk et al. (2011) rediscovered these sets in the case of *homogeneous atoms* in the context of *automata theory* and called them *orbit-finite sets with atoms*.

4. Up to *isomorphism*, a structure is *hereditarily definable* ⇔ it *interprets* in Atoms

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