Can Modalities Save Naive Set Theory?

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To the memory of Prof. Grigori Mints, Stanford University
Born: June 7, 1939, St. Petersburg, Russia
Died: May 29, 2014, Palo Alto, California

1 Background

In October of 2009 at Stanford University, the late Grigori "Grisha" Mints asked the senior author whether a naive set theory could be consistent in modal logic. Here are two versions of a modal comprehension principle:

$$(\exists y)(\forall x)(x \in y \leftrightarrow \Box\varphi) \qquad \text{(Comp}\Box\text{)}$$

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\varphi) \qquad (\Box\text{Comp}\Box)$$

where as usual y is not free in φ. The idea of the question is that by restricting to modalized formulas the derivation of the usual Russell Paradox could be blocked.

In the most commonly used systems, where the Converse Barcan Formula ($\Box\forall x\varphi \rightarrow \forall x\Box\varphi$) is derivable, ($\Box$Comp$\Box$) follows from the principle

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \Box\varphi).$$

(If the Barcan Formula ($\forall x\Box\varphi \rightarrow \Box\forall x\varphi$) is also a theorem schema, the two comprehension principles are equivalent.) In his lecture, later revised for 2010, Scott presented his Modal ZF, which uses the stronger pattern of modal operators, but comprehension is restricted by a membership clause as is usually done in standard ZF:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow x \in u \land \varphi). \qquad \text{(MZF Comp)}$$

Scott was working in the Lewis system **S4** of modal logic and Mints was happy to position his question in the same modal system. Obviously a very, very weak modality can avoid paradoxes, but such results may not be especially interesting.

At that time Scott could not answer the consistency question, and neither could Mints, though they both agreed that a set theory based on (Comp$\Box$) would probably be very weak. And there, to the best of our knowledge, the problem sat ever since.

Last November Scott received a notice from Carnegie Mellon that there would be a philosophy seminar on a naive set theory by Lederman (see Field *et al.* (forthcoming)). Scott wrote him for his paper and said, "By the way, there is this question of Grisha Mints, and I wonder if you have an opinion?" Lederman sent back a sketch of a proof of inconsistency for the strengthened version of ($\Box$Comp$\Box$), which did not quite work out, but the exchange became the basis for sections 3-6 of the present note.

In the first draft of the paper, Scott and Lederman left open the consistency of (Comp$\Box$), although they observed that it was not inconsistent by the analogue of the Russell set alone. Scott and Lederman tried out several model possibilities for the consistency of that principle, without success. In March of 2015 Liu approached them with a related model, which after a small correction gave a consistency proof. A few days later, Fritz approached them with essentially the same model, and his presentation is the basis of section 2. Fritz later proved the results in section 7.1, and provided the discussion of related work.

Modalized comprehension principles have been studied in a number of different settings in the literature. One is intensional higher-order logic (see, e.g., Gallin (1975, p. 77) or Zalta (1988, p. 22)), where a syntactic distinction between types allows for an unrestricted comprehension principle. Such discussions usually work with models with constant (first- and higher-order) domains; for discussions of comprehension principles appropriate for variable domains of all types, see Williamson (2013, chapter 6.3–6.4) and Fritz & Goodman (unpublished, section 5).

Another common form of modal comprehension principles occurs in modal set theories which are obtained by modalizing common set theories (e.g., such a system for metaphysical necessity is presented in Fine (1981)). Systems for epistemic modalities were developed by several authors in the 1980s (e.g. the contributions by Myhill, Goodman and Ščedrov to Shapiro (1985), or the references in Goodman (1990)). In such theories, comprehension is usually restricted as in (MZFComp) above.

Both of these kinds of modal comprehension principles differ fundamentally from the naive principles in that they are modalizations of versions of comprehension which are already consistent. Modalizing naive comprehension in order to make it consistent has been less widespread, but several such strands can be identified in the literature. The first uses modality to make the iterative conception of set explicit by reformulating comprehension to say that at some stage, there is a set defined by a given condition, using a possibility operator to formalize "at some stage". Pioneered by Parsons (1983), such principles were recently investigated in Studd (2013) and Linnebo (2013); see also Linnebo (2010).

The second strand goes back to Aczel & Feferman (1980), who save the naive comprehension principle from inconsistency by replacing its material biconditional by an intensional one; see Feferman (1984) for a survey of related literature. Even closer to ($\Box$Comp$\Box$) is the following comprehension principle proposed by Krajíček (1987):

$$(\exists y)(\forall x)((\Box x \in y \leftrightarrow \Box\varphi) \wedge (\Box\neg x \in y \leftrightarrow \Box\neg\varphi)) \qquad \text{(MCA)}$$

Krajíček proves that this principle is inconsistent in **S5**, and it seems still to be an open problem whether it is consistent in the relatively weak modal logic **T** (see Krajíček (1988) and Kaye (1993)).

Finally, Fitch (1966) proposed a comprehension principle which may be rendered as (Comp$\Box$). Fitch motivates this principle philosophically in Fitch (1967b), and formally develops a set theory on the basis of it in Fitch (1967a) (a correction, prompted by a review of Rundle (1969), appeared as Fitch (1970)). This formal development is in the form of a combinatory logic, rather than a standard first-order modal logic, which makes a comparison to the present proposal difficult; see (Cantini, 2009, section 4.2) for helpful discussion.

(Comp$\Box$) is perhaps the most natural way of using a modality to restrict naive comprehension, so we shall begin by investigating a set theory based on this principle. We find that it is consistent, but the model we provide shows that it is too weak to provide a basis for mathematics beyond number theory. We thus study ($\Box$Comp$\Box$) in the remainder of the paper, and find that in common modal systems an inconsistency can be derived by a suitable version of the Russell paradox.

2 The Consistency of (Comp$\Box$)

Our language will be the language of predicate logic with $\neg, \wedge, \forall$ plus identity $=$ and the relation symbol $\in$, along with the unary modal operator $\Box$. The symbols $\vee, \rightarrow, \exists$ and $\Diamond$ are introduced as metalinguistic abbreviations in the usual way. The modal system **T** can then be axiomatized with the following schematic axioms and rules:

(LPC)	Any substitution instance of a theorem of predicate logic.
(K)	$\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(T)	$\vdash \Box\varphi \rightarrow \varphi$
(MP)	From $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ infer $\vdash \psi$
($\forall$2)	From $\vdash \varphi \rightarrow \psi$ infer $\vdash \varphi \rightarrow \forall x\psi$, provided x is not free in φ
(RN)	From $\vdash \varphi$ infer $\vdash \Box\varphi$

Using (K), (MP), and (RN), it is routine to show that **T** has the derived rule

(RM) From $\vdash \varphi \rightarrow \psi$ infer $\vdash \Box\varphi \rightarrow \Box\psi$ and $\vdash \Diamond\varphi \rightarrow \Diamond\psi$

This system does not allow the derivation of the Barcan formula ($\forall x \Box \varphi \to \Box \forall x \varphi$). It does, however, allow us to derive the converse Barcan formula ($\Box \forall x \varphi \to \forall x \Box \varphi$).[1] Since every theorem schema of propositional logic is a theorem schema of predicate logic, the above logic contains propositional logic. We will often write "PL" for steps in the proofs which are justified by invoking obvious propositional theorems, together with modus ponens. The logic **T**, and all logics we discuss in this paper allow the replacement of provable equivalents in all contexts; we refer to this rule as (Rep).

S4 is the system which results by adding the schematic axiom (L4) $\Box \varphi \to \Box \Box \varphi$ to the above axiomatization of **T** (and closing under the rules). **S5** is the system which results by adding (L5) $\neg \Box \varphi \to \Box \neg \Box \varphi$ to **T**; the resulting system proves every instance of (L4). In the quantified setting as above, **S5** (and any logic which contains the modal axiom (B) ($\varphi \to \Box \Diamond \varphi$) proves every instance of the Barcan Formula. **T** is a sublogic of **S4** and both are sublogics of **S5**, the system of this section.

2.1 Consistency

To construct a model validating (Comp$\Box$), fix any countably infinite set D; this will serve as the (constant) domain of the model. Assuming identity is always necessary, and since any finite or cofinite set X can be specified using only identity and a finite number of parameters, the model must interpret $\in$ in such a way that at each world, there is an element which contains all and only the members of X. The idea behind the following model is to let the interpretation of $\in$ vary sufficiently among worlds so that no further witnesses for (Comp$\Box$) are required.

Thus, letting F be the set of finite and co-finite subsets of D, we use the bijections $w : D \leftrightarrow F$ from D to F as the worlds, and interpret $\in$ using the following interpretation function:

$$V(\in, w) = \{\langle o_1, o_2 \rangle \in D \times D |\ o_1 \in w(o_2)\}$$

The simplest variant of this model construction would be to take as our set of worlds simply the set of *all* bijections from D to F, but this set would be uncountable. To show that we can make do with a countable set of worlds, we proceed as follows.

For any permutation π of D and bijection $f : D \to F$, define $\pi(f) : D \to F$ such that $\pi(f)(\pi(o)) = \pi(f(o))$ for all $o \in D$. In such a case, $\pi(f)$ is a bijection from D to F as well. For any permutation π of D, let the set of elements of D not mapped to themselves by π be called the *support* of π; let S be the set of permutations of D whose support is finite. Choosing any bijection $b : D \to F$, we can now construct the set of worlds as follows: $W = \{\pi(b) | \pi \in S\}$. Since D is countable, S and W are countable as well. Further, since S is closed under composition, $\pi(w) \in W$ for all $\pi \in S$ and $w \in W$. Note also that $b \in W$.

Let the model $M = \langle W, D, V \rangle$, and define truth of a formula φ relative to M, a world $w \in W$

[1] For a proof, see section 7.1, where we extend some of our main results to a logic in which this principle cannot be derived.

and assignment function a from the set of variables to D (written $M, w, a \vDash \varphi$) as usual. Since there is no accessibility relation, $\Box$ is interpreted as truth in all worlds, and since there is no varying domain function, quantifiers range over D at all worlds.

To show that M validates (Comp$\Box$), we start with a few definitions. First, the extension of a formula with a distinguished variable relative to a world and assignment:

$$[\![\varphi(x)]\!]_{M,w,a} = \{o \in D | M, w, a[o/x] \vDash \varphi\}.$$

Next, if π is any permutation of D, we extend applying π to two further constructions in natural ways: For any $O \subseteq D$, let $\pi(O) = \{\pi(o) | o \in O\}$. For any assignment function a, let $\pi(a)$ be the assignment function such that $\pi(a)(z) = \pi(a(z))$. In the following lemmas, unless noted otherwise, π is an arbitrary member of S.

Lemma 2.1. *$M, w, a \vDash \varphi$ iff $M, \pi(w), \pi(a) \vDash \varphi$.*

Proof. By induction on the complexity of φ; only the case for $\in$ is interesting:
$M, w, a \vDash x \in y$ iff
$a(x) \in w(a(y))$ iff
$\pi(a(x)) \in \pi(w(a(y)))$ iff
$\pi(a(x)) \in \pi(w)(\pi(a(y)))$ iff
$\pi(a)(x) \in \pi(w)(\pi(a)(y))$ iff
$M, \pi(w), \pi(a) \vDash \varphi$. QED

Lemma 2.2. $\pi([\![\varphi(x)]\!]_{M,w,a}) = [\![\varphi(x)]\!]_{M,\pi(w),\pi(a)}$.

Proof. $\pi([\![\varphi(x)]\!]_{M,w,a})$
$= \{\pi(o) | o \in D \text{ and } M, w, a[o/x] \vDash \varphi\}$
$= \{o \in D | M, w, a[\pi^{-1}(o)/x] \vDash \varphi\}$
$= \{o \in D | M, \pi(w), \pi(a[\pi^{-1}(o)/x]) \vDash \varphi\}$ (by the previous lemma)
$= \{o \in D | M, \pi(w), \pi(a)[o/x] \vDash \varphi\}$
$= [\![\varphi(x)]\!]_{M,\pi(w),\pi(a)}$ QED

Lemma 2.3. *If $\pi(a(z)) = a(z)$ for all variables z free in φ, $\pi([\![\Box\varphi(x)]\!]_{M,w,a}) = [\![\Box\varphi(x)]\!]_{M,w,a}$.*

Proof. $\pi([\![\Box\varphi(x)]\!]_{M,w,a})$
$= [\![\Box\varphi(x)]\!]_{M,\pi(w),\pi(a)}$ (by the previous lemma)
$= \bigcap_{v \in W} [\![\varphi(x)]\!]_{M,v,\pi(a)}$
$= \bigcap_{v \in W} [\![\varphi(x)]\!]_{M,v,a}$ (since $\pi(a(z)) = a(z)$ for all variables z free in φ)
$= [\![\Box\varphi(x)]\!]_{M,w,a}$ QED

Lemma 2.4. *If $O \subseteq D$ is finite and $O' \subseteq D$ is such that $\pi(O') = O'$ for all $\pi \in S$ such that $\pi(o) = o$ for all $o \in O$, then $O' \in F$.*

Proof. Assume for the sake of contradiction that $O' \notin F$. Then there are $o_1, o_2 \in D \backslash O$ such that $o_1 \in O'$ and $o_2 \notin O'$. Now consider the transposition $(o_1 o_2)$ which switches o_1 and o_2. $(o_1 o_2) \in S$, but $(o_1 o_2)(O') \neq O'$. But this contradicts the assumption. QED

Theorem 2.5. (Comp□) *is valid in the model M.*

Proof. Consider any φ in which y is not free, $w \in W$ and assignment a. It suffices to show that $M, w, a \vDash \exists y \forall x (x \in y \leftrightarrow \Box \varphi)$. Note that by the preceding two lemmas, $[\![\Box \varphi(x)]\!]_{M,w,a} \in F$. By construction of M, there is an $o \in D$ such that $w(o) = [\![\Box \varphi(x)]\!]_{M,w,a}$; this witnesses the existential claim. QED

Proposition 2.6. *Aside from the principles of* **S5**, *the following are valid in the model:*

$$
\begin{array}{rl}
\text{(Bar)} & \Box \forall x \varphi(x) \leftrightarrow \forall x \Box \varphi(x) \\
\text{(Ext)} & (\forall y)(\forall z)\left[(\forall x)\left(x \in y \leftrightarrow x \in z\right) \rightarrow y = z\right] \\
\text{(Neg)} & (\forall z)(\exists y)(\forall x)\left[x \in y \leftrightarrow \neg(x \in z)\right] \\
\text{(Con)} & (\forall z_1)(\forall z_2)(\exists y)(\forall x)\left[x \in y \leftrightarrow (x \in z_1 \wedge x \in z_2)\right] \\
\text{(Comp}\Diamond\text{)} & (\exists y)(\forall x)\left[x \in y \leftrightarrow \Diamond \varphi(x)\right] \\
\text{(Equ)} & \forall x \forall y (\Diamond x = y \rightarrow \Box x = y) \\
\text{(Mem)} & \forall x \forall y \Diamond x \in y \\
\text{(Non)} & \forall x \forall y \Diamond \neg x \in y
\end{array}
$$

Proof. All but the last two are straightforward. For (Mem), first, we have to show that for any a, there is some $\pi(b) \in W$ satisfies $x \in y$.

Case 1. $a(x) = a(y)$. Let π be a permutation with finite support such that for some $i \in \mathbb{N}$, $i \in \pi(b)(i)$. Then pick a finite-support π^* with $\pi^*(b)(i) = a(x)$, and let $w = \pi^*(\pi(b))$ guaranteeing $a(x) \in \pi(b)(a(x))$. Thus $M, \pi^*(\pi(b)), a \vDash x \in y$.

Case 2. $a(x) \neq a(y)$. Pick distinct $i, j \in \mathbb{N}$, and a π such that $i \in \pi(b)(j)$ holds. Now define $\pi^*(b)$ such that $\pi^*(b)(i) = a(x)$ and $\pi^*(b)(j) = a(y)$ so that $M, \pi^*(\pi(b)), a \vDash x \in y$.

The proof of (Non) is similar.

QED

2.2 Undecidability

We rely on the following general result of Tarksi, Mostowski and Robinson (1953, pg. 18, Theorem 6):

Theorem. *Let T_1 and T_2 be two compatible theories such that every constant of T_2 is a constant of T_1. If T_2 is essentially undecidable and finitely axiomatizable, then T_1 is undecidable, and so is every sub theory of T_1 which has the same constants.*

Call a binary relation R on a countably infinite set D "memberly" if the transformation $m \mapsto \{n|nRm\}$ is a bijection between D and F (where again F is the set of finite and cofinite subsets of D). Letting a memberly structure be an infinite set together with a memberly relation on the set, we show that one can interpret Robinson Arithmetic in such memberly structures. Since a memberly structure induces a bijection $b : D \to F$, memberly structures can be used as the basis for models of (Comp□)+**S5** such as the one constructed for Theorem 2.5. Moreover, since Robinson Arithmetic is known to be essentially undecidable and finitely axiomatizable, our interpretation of Robinson Arithmetic on memberly structures will demonstrate that (Comp□)+**S5** is consistent with an essentially undecidable, finitely axiomatizable theory which contains the same constants, and hence that (Comp□)+**S5** is itself undecidable.

To provide an interpretation of Robinson Arithmetic, we use the fact that "Adjunctive Set Theory" is known to interpret Robinson Arithmetic (Montagna & Mancini (1994)). All we must show is that the following two axioms of this theory can be interpreted on our structures:

$$\exists y \forall x (\neg x \in y) \tag{Empty}$$

$$\forall y \forall z \exists w \forall x (x \in w \leftrightarrow x \in y \vee x = z) \tag{Add}$$

To interpret *this* theory, we make use of the possibility of defining finiteness in memberly structures. Clearly if this can be done, then restricting attention to the finite sets, we can show that the finite sets provide a model of the axiom (Add). Given the necessity of identity, (Comp□) entails (Empty), as can be seen by instantiating on $x \neq x$.

For clarity, we use $\in$ as a symbol for the memberly relation R. Among the finite sets given by the memberly relation are all singletons; the singleton of a can be defined as $\{a\} = \{x|x = a\}$. But given that exactly the finite and cofinite sets exist, it is readily seen that a is finite iff $\{\{x\}|x \in a\}$ exists. We can use this fact to define finiteness. This allows us to show the satisfaction of the axiom (Add) on the finite sets, which shows by the paper already cited that Robinson Arithmetic can eventually be interpreted in memberly structures.

In fact, we can also interpret Robinson Arithmetic by a more direct method. In addition to the singletons, among the finite sets given by the memberly relations are pairs, which can be defined by: $\{a, b\} = \{x|x = a \vee x = b\}$. Together with the definability of singletons, this allows us to define ordered pairs as $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$, which in turn allows us to define equivalence of cardinality among the finite sets in the usual way. Given the notion of cardinal equivalence we make use of the standard definitions of union and cartesian products to give an interpretation of arithmetic.

Either method leads to the result that memberly structures allow for the interpretation of an essentially undecidable theory, and thus shows that (Comp□)+**S5** is itself undecidable.

2.3 Concluding Discussion

The model shows that (Comp$\Box$) gives rise only to a very weak set theory, since it is consistent with the only sets being the finite and cofinite ones. The weakness of the set theory is naturally attributed to the fact that membership is not a matter of necessity. In the model above, for any pair of sets, it is possible that the one be a member of the other.

A direct strategy for resolving this problem would be to impose the requirement that membership be necessary, that is, to add the axiom $x \in y \rightarrow \Box x \in y$. But this leads immediately to inconsistency: the instance for $\varphi = x \notin x$ is easily seen to be inconsistent in the very weak normal modal logic **KD**, axiomatized by replacing the axiom schema (T) in the modal logic **T** with the schema (D): $\Box\varphi \rightarrow \Diamond\varphi$. (A similar result is obtained if one adds "hybrid" operators such as "actually" @; this leads to inconsistency no matter the logic of $\Box$, since it allows us to define the standard Russell set by instantiating (Comp$\Box$) on $\Box @ \neg x \in x$.)

An apparently more promising approach is to use ($\Box$Comp$\Box$), which strengthens the material biconditional in (Comp$\Box$) to a strict biconditional. This principle was the subject of Mints's original question, and it will be the focus of the remainder of the paper.

3 Inconsistency of ($\Box$Comp$\Box$)

Our first inconsistency result will use the following instance of ($\Box$Comp$\Box$):

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\neg x \in x) \qquad (\Box\text{Russell}\Box)$$

Theorem 3.1. *($\Box$Russell$\Box$) is inconsistent in* **T**.

Proof.

(1)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \rightarrow \Box(R \in R \leftrightarrow \Box\neg R \in R)$	Universal Instantiation
(2)	$\Box(R \in R \leftrightarrow \Box\neg R \in R) \rightarrow (R \in R \leftrightarrow \Box\neg R \in R)$	(T)
(3)	$(R \in R \leftrightarrow \Box\neg R \in R) \rightarrow (R \in R \rightarrow \neg R \in R)$	(T)
(4)	$(R \in R \leftrightarrow \Box\neg x \in x) \rightarrow \neg R \in R$	2, 3, PL
(5)	$\Box(R \in R \leftrightarrow \Box\neg R \in R) \rightarrow \Box\neg R \in R$	4, RM
(6)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \rightarrow (R \in R \leftrightarrow \Box\neg R \in R)$	2, 3
(7)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \rightarrow (\Box\neg R \in R \wedge (R \in R \leftrightarrow \Box\neg R \in R))$	5, 6, PL
(8)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \rightarrow (\Box\neg R \in R \wedge R \in R)$	7, PL
(9)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \rightarrow (\neg R \in R \wedge R \in R)$	8, (T), PL
(10)	$\neg(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x)$	9, PL
(11)	$\forall R \neg(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x)$	Universal Generalization, 10

But (8) contradicts ($\Box$Russell$\Box$). QED

In fact, Theorem 3.1 does not depend on special laws for the quantifier, as it may be seen as an instance of the following general fact about the propositional fragment of the logic:

Proposition 3.2. *If* $\vdash \varphi \to (\psi \leftrightarrow \Box\neg\psi)$ *then* $\vdash \neg\Box\varphi$ *in* **T**.

Proof.

(1)	$\varphi \to (\psi \leftrightarrow \Box\neg\psi)$	(Assumption)
(2)	$(\psi \leftrightarrow \Box\neg\psi) \to (\psi \to \neg\psi)$	T
(3)	$\varphi \to \neg\psi$	1, 2, PL
(4)	$\Box\varphi \to \Box\neg\psi$	3, RM
(5)	$\Box\varphi \to (\psi \leftrightarrow \Box\neg\psi)$	1, T
(6)	$\Box\varphi \to (\Box\neg\psi \wedge (\psi \leftrightarrow \Box\neg\psi))$	4, 5, PL
(7)	$\Box\varphi \to (\Box\neg\psi \wedge \psi)$	6, PL
(8)	$\Box\varphi \to (\neg\psi \wedge \psi)$	7, T, PL
(9)	$\neg\Box\varphi$	8, PL

QED

Letting $(R \in R \leftrightarrow \Box\neg R \in R) = \varphi$ and $R \in R = \psi$, we derive the contradiction for (□Comp□) as follows:

(10)	$\neg\Box(R \in R \leftrightarrow \Box\neg R \in R)$	Proposition 3.2
(11)	$\exists x\neg\Box(x \in R \leftrightarrow \Box\neg x \in x)$	10, Existential Introduction
(12)	$\neg\forall x\Box(x \in R \leftrightarrow \Box\neg x \in x)$	10, 11, Df∀
(13)	$\forall y\neg\forall x\Box(x \in y \leftrightarrow \Box\neg x \in x)$	12, Universal Generalization
(14)	$\neg\exists y\forall x\Box(x \in y \leftrightarrow \Box\neg x \in x)$	13, Df∀

In later proofs, we will provide the argument for the propositional fragment only; the steps for introducing the quantifiers will be the same as above.

Note also that if we work in **S4** and **S5**, Proposition 3.2 can be strengthened to:

Proposition 3.3. *If* $\vdash \Box\varphi \to (\psi \leftrightarrow \Box\neg\psi)$ *then* $\vdash \neg\Box\varphi$ *in* **S4**.

The reason is that using (RM), (T), (L4) and (Rep), one can derive the following rule in **S4** (and hence **S5**):

$$\text{From } \vdash \Box\varphi \to \psi \text{ infer } \vdash \Box\varphi \to \Box\psi. \qquad (\Box\ \text{RM})$$

If we start from the assumption $\Box\varphi \to (\psi \leftrightarrow \Box\neg\psi)$, in the above argument, at step (3) we obtain $\Box\varphi \to \neg\psi$. In **T** this is weaker than (3) above, and is a dead end. But in **S4** and **S5**, we use (□RM) to obtain (4) as above, and the remainder of the proof follows as before. This proposition will play an important role in the next sections where we reformulate (□Comp□) using a different modality than □ before the instance of φ.

4 Inconsistency of $(\Box\text{Comp}\Box\Diamond)$

Next we ask whether the following variant of $(\Box\text{Comp}\Box)$

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\varphi) \qquad (\Box\text{Comp}\Box\Diamond)$$

can be consistently added to **S4**. We suspect Mints himself would have asked this and related questions, had he seen the weakness of $(\text{Comp}\Box)$ and our first contradiction using $(\Box\text{Comp}\Box)$. But in fact this new principle is also inconsistent, as we now show, once again by considering an instance:

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\neg x \in x) \qquad (\Box\text{Russell}\Box\Diamond)$$

Theorem 4.1. $(\Box\text{Russell}\Box\Diamond)$ *is inconsistent in* **S4***.*

Again, the theorem will follow from a proposition provable in the propositional fragment, although this time the proof uses the full strength of **S4**. For note that the following is a law of **S4** (and hence **S5**):

$$\Box(\varphi \leftrightarrow \psi) \rightarrow (\chi \leftrightarrow \chi') \qquad (\text{Rep}\Box)$$

where χ is like χ' except that in the latter 0 or more instances of φ have been replaced by ψ, and in no instances of φ or ψ within χ or χ' are any free variables in those formulas bound. This differs from ordinary (Rep), in which replacement in all contexts is licensed by the provable equivalence of φ and ψ.

Proposition 4.2. *If* $\vdash \Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\neg\psi)$ *then* $\vdash \neg\Box\varphi$ *in* **S4***.*

Proof.

(1)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\neg\psi)$	(Assumption)
(2)	$\Box\varphi \rightarrow \Box(\psi \leftrightarrow \Box\Diamond\neg\psi)$	1, $\Box$RM
(3)	$\Box(\psi \leftrightarrow \Box\Diamond\neg\psi) \rightarrow (\Box\psi \leftrightarrow \Box\Box\Diamond\neg\psi)$	K, PL
(4)	$\Box\Box\Diamond\neg\psi \leftrightarrow \Box\Diamond\neg\psi$	T, L4
(5)	$(\Box\psi \leftrightarrow \Box\Box\Diamond\neg\psi) \leftrightarrow (\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	4, Rep
(6)	$\Box\varphi \rightarrow (\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	2, 3, 4, 5
(7)	$\Box\varphi \rightarrow \Box(\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	6, $\Box$RM
(8)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\psi)$	1, 7, Rep$\Box$
(9)	$\Box\varphi \rightarrow \Box(\psi \leftrightarrow \Box\psi)$	8, $\Box$RM
(10)	$\Box(\psi \leftrightarrow \Box\psi) \leftrightarrow \Box(\neg\psi \leftrightarrow \Diamond\neg\psi)$	PL, Df$\Diamond$
(11)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\neg\psi)$	1, 9, 10, Rep$\Box$
(12)	$\neg\Box\varphi$	11, Proposition 3.3

QED

Theorem 4.1 now follows immediately.

5 Inconsistency of $(\Box\text{Comp}\Box\Diamond\Box)$

Consider next whether the following further variant on (□Comp□)

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\Box\varphi) \qquad (\Box\text{Comp}\Box\Diamond\Box)$$

can be consistently added to **S4**. Once again, we show that it cannot be.

We use the following instance of (□Comp□◇□):

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \Box\Diamond\Box\neg x \in x). \qquad (\Box\text{Russell}\Box\Diamond\Box)$$

Theorem 5.1. $(\Box\text{Russell}\Box\Diamond\Box)$ *is inconsistent in* **S4**.

Recall that **S4** proves every instance of the following "reduction law":

$$\Box\Diamond\varphi \leftrightarrow \Box\Diamond\Box\Diamond\varphi. \qquad (\text{Red}\Box\Diamond)$$

The theorem is a consequence of the following proposition.

Proposition 5.2. *If* $\vdash \Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$ *then* $\vdash \neg\Box\varphi$ *in* **S4**.

Proof.

(1)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$	(Assumption)
(2)	$\Box\varphi \rightarrow \Box(\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$	1, □RM
(3)	$\Box(\psi \leftrightarrow \Box\Diamond\Box\neg\psi) \rightarrow (\Box\psi \leftrightarrow \Box\Box\Diamond\Box\neg\psi)$	K, PL
(4)	$\Box\Box\Diamond\Box\neg\psi \leftrightarrow \Box\Diamond\Box\neg\psi$	T, L4
(5)	$(\Box\psi \leftrightarrow \Box\Box\Diamond\Box\neg\psi) \leftrightarrow (\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$	4, Rep
(6)	$\Box\varphi \rightarrow (\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$	2, 3, 4, 5
(7)	$\Box\varphi \rightarrow \Box(\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$	6, □RM
(8)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\psi)$	1, 7, Rep□
(9)	$\Box\varphi \rightarrow \Box(\psi \leftrightarrow \Box\psi)$	8, □RM
(10)	$\Box(\psi \leftrightarrow \Box\psi) \leftrightarrow \Box(\neg\psi \leftrightarrow \Diamond\neg\psi)$	PL, Df◇
(11)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\Box\Diamond\neg\psi)$	1, 9, 10, Rep□
(12)	$\Box\varphi \rightarrow (\psi \leftrightarrow \Box\Diamond\neg\psi)$	11, Red□◇
(13)	$\neg\Box\varphi$	12, Proposition 4.2

QED

Theorem 5.1 now follows immediately.

6 Duality Between Modalities

S4 is known to have fourteen modalities (see, e.g., Chellas 1980 for proofs), which we can divide into positive and negative, as follows:

Positive	Negative
$\neg\neg$	$\neg$
$\Box$	$\Box\neg$
$\Diamond$	$\Diamond\neg$
$\Box\Diamond$	$\Box\Diamond\neg$
$\Diamond\Box$	$\Diamond\Box\neg$
$\Box\Diamond\Box$	$\Box\Diamond\Box\neg$
$\Diamond\Box\Diamond$	$\Diamond\Box\Diamond\neg$

Starting from the pattern exhibited by our three contradictions, one might wonder whether, given a modality M in the above list, (□RussellM) is guaranteed to be inconsistent, but that is not so. For any of the negative modalities, (□RussellM) is in fact consistent, as can be shown by the one-element, one-world model in which R is the only element of the domain, $R \in R$ is true at the unique world, and the accessibility relation is universal.

But this does not show that the more powerful principle (□CompM) are consistent for any of the negative modalities or for the as-yet-untouched positive ones. In fact, it is clear that if for a positive modality M, (□RussellM) is inconsistent, then since (□RussellM) is an instance of (□CompM¬), that principle must also be inconsistent. (□Comp¬¬) entails naive comprehension by the modal axiom (T) and are inconsistent by Russell's original paradox. The second, fourth, and sixth of the above positive modalities were shown to be inconsistent in **S4** by Theorems 3.1, 4.1, and 5.1, respectively. Thus for each of the corresponding negative modalities M, (□CompM) is also inconsistent.

We now turn to the remaining positive ones. Here we observe that if for a modality M, (□CompM) can be shown to be inconsistent by instantiating on an atomic formula or its negation φ, then (□Comp¬M¬) is also inconsistent. For suppose we have a proof that an instance of (□CompM) leads to inconsistency using a formula φ. Then consider the instance of (□Comp¬M¬) which uses the same φ, that is (moving negations):

$$(\exists y)(\forall x)\Box(\neg x \in y \leftrightarrow M\neg\varphi) \qquad \text{(□CompDual)}$$

Define $x \notin y$ by $x \notin y \leftrightarrow \neg x \in y$, and write φ^{-1} for the formula resulting from replacing $\in$ in φ with $\notin$. Since φ is either atomic or the negation of an atomic sentence, we have $\vdash \varphi \leftrightarrow \neg\varphi^{-1}$, licensing intersubstitution in all contexts. Thus (□CompDual) can be written:

$$(\exists y)\Box(\forall x)(x \notin y \leftrightarrow M\varphi^{-1}). \qquad \text{(□CompDual+)}$$

But given that the relevant instance of (□CompM) led to inconsistency for $\in$, (□CompDual+) will lead to inconsistency by the same argument applied to $\notin$.

Since the remaining three positive modalities are simply the duals of those shown to be inconsistent by Theorems 3.1, 4.1, 5.1, this observation allows us to put an end to one line of questioning. If M is any of the fourteen standard modalities in **S4**, then ($\Box$CompM) is inconsistent. And if M is one of the seven positive modalities, then ($\Box$RussellM) is inconsistent.

7 Conclusion

We now turn to three proposals for future work: one, based on logics which do not validate the converse Barcan formula; a second based on replacing the modal axiom schema (T) with the weaker (D); and a third, based on the Gödel-McKinsey-Tarski translations of naive comprehension into a modal language.

7.1 Converse Barcan Formula

As we noted in Section 2, the proof system stated above allows us to derive the Converse Barcan Formula (CBF):

(1)	$\forall x\varphi \rightarrow \varphi$	predicate logic
(2)	$\Box\forall x\varphi \rightarrow \Box\varphi$	(RN), (K)
(3)	$\Box\forall x\varphi \rightarrow \forall x\Box\varphi$	$(\forall 2)$

In the Introduction, we noted that CBF can be used in deriving ($\Box$CompM) from the principle:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \Box\varphi). \qquad (\Box\forall\text{Comp}\Box)$$

Without the CBF, the principles may have incommensurable strength. Thus it is natural to wonder whether our inconsistencies for ($\Box$CompM) depends on the derivability of CBF. In fact, as we now show, the results for ($\Box$CompM) do not. Interestingly, however, there are still open questions concerning the consistency of ($\Box\forall$CompM) for some modalities M.

We use the proof system of (Hughes & Cresswell, 1996, Ch. 16), letting Ex be an abbreviation for $\exists y(y = x)$ (the "existence predicate") (for further discussion see Scott (1967) and Scott (1979)). Crucially, universal instantiation is restricted as follows:

$$\forall x\varphi \rightarrow (Ey \rightarrow \varphi[y/x]), \qquad \text{(RestGen)}$$

Call the resulting system "**Free T**" (and similarly for extensions of **T**).

Note that the following laws and rule can still be derived in this system:

$$\forall x Ex \qquad \text{(UE)}$$

$$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi) \qquad (\forall \rightarrow)$$

$$\text{If } \vdash \varphi \text{ then } \vdash \forall x\varphi \qquad \text{(UG)}$$

We first show that if M is a modality of **S4**, ($\Box$CompM) is inconsistent:

Theorem 7.1. *If M is any positive modality of* **S4***, then* (□RussellM) *is inconsistent in* **Free S4**.

Proof.

(1)	$\neg\Box(R \in R \leftrightarrow M\neg R \in R)$	Propositions 3.2, 4.2, 5.2, Duality
(2)	$\forall x\Box(x \in R \leftrightarrow M\neg x \in x) \rightarrow (ER \rightarrow \Box(R \in R \leftrightarrow M\neg R \in R))$	(RestGen)
(3)	$ER \rightarrow \neg\forall x\Box(x \in R \leftrightarrow M\neg x \in x)$	1, 2, PL
(4)	$\forall y(Ey \rightarrow \neg\forall x\Box(x \in y \leftrightarrow M\neg x \in x))$	3, UG
(5)	$\forall yEy$	UE
(6)	$\forall y\neg\forall x\Box(x \in y \leftrightarrow M\neg x \in x)$	4, 5, $\forall\rightarrow$, PL
(7)	$\neg\exists y\forall x\Box(x \in y \leftrightarrow M\neg x \in x)$	6, Df$\forall$

which contradicts (□CompM). Inconsistency for the negative modalities follows from the observations of the previous section. QED

The above argument can also be used to show that (□Comp□) is inconsistent in **Free T**, by invoking Proposition 3.2 in step (1).

Next, we show that (□∀Comp□) is inconsistent in **Free T**, by considering the following instance of that principle:

$$\exists y\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)). \qquad (\Box\forall\text{RussellE}\Box)$$

Theorem 7.2. (□∀RussellE□) *is inconsistent in* **Free T**.

Proof.

(1)	$\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow (ER \rightarrow (R \in R \leftrightarrow \Box(ER \rightarrow R \notin R)))$	RestGen
(2)	$(R \in R \leftrightarrow \Box(ER \rightarrow R \notin R)) \rightarrow (R \in R \rightarrow (ER \rightarrow R \notin R))$	(T)
(3)	$(R \in R \rightarrow (ER \rightarrow R \notin R)) \rightarrow (ER \rightarrow R \notin R)$	PL
(4)	$\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow (ER \rightarrow R \notin R)$	1-3
(5)	$\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow \Box(ER \rightarrow R \notin R)$	4
(6)	$\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow (ER \rightarrow (R \in R \leftrightarrow \Box(ER \rightarrow R \notin R)))$	1, (T)
(7)	$\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow (\Box(ER \rightarrow R \notin R) \wedge (ER \rightarrow R \in R))$	5, 6
(8)	$\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow ((ER \rightarrow R \notin R) \wedge (ER \rightarrow R \in R))$	7
(9)	$\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)) \rightarrow \neg ER$	8
(9)	$\forall R(ER \rightarrow \neg\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x)))$	9, UG
(10)	$\forall RER$	UE
(11)	$\forall R\neg\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x))$	9, 10, ($\forall\rightarrow$)
(12)	$\neg\exists R\Box\forall x(x \in R \leftrightarrow \Box(Ex \rightarrow x \notin x))$	11

QED

S5 proves every instance of the schema $\Box\varphi \leftrightarrow \Diamond\Box\varphi$; hence there is an instance of (□∀Comp◇□) which is provably equivalent to (□∀RussellE□), and thus for all positive modalities in **Free S5**,

($\Box\forall$CompM) is inconsistent. We leave the consistency of ($\Box\forall$CompM) for the modalities in **Free S4** other than $\Box$ for future work.

7.2 Replacing (T) with (D)

The results of this paper have been confined to logics which extend the modal system **T**. It is natural to wonder whether the impossibility results can be avoided by moving to systems in which (T) is weakened. Systems which do not contain **KD** are of little interest, since they are valid in "dead end" worlds and the model with a single such world together with a domain which contains a universal set is a model of both ($\Box$Comp$\Box$) and (Comp$\Box$); thus these principles are trivially consistent in these logics.

A more promising approach is to replace (T) by (D) ($\Box\varphi \to \Diamond\varphi$). This weakening of the logic is suggested by a particular "fictionalist" approach to the philosophy of mathematics. Motivated by the idea that ZF and other foundational theories of mathematics commit themselves to an objectionably Platonist ontology, some have argued that mathematics should be seen as a particular kind of fiction, where the fiction is understood to contain anything deducible from what is already part of the fiction. We may formalize this theory using our modal language, by interpreting $\Box$ as "according to the fiction" or "it is true according to the fiction that". Since fictions need not be true, the principle (T) is inappropriate to this interpretation. So one might wonder whether a fictionalist of this kind can provide the foundations for mathematics using a naive comprehension principle along the lines of ($\Box$Comp$\Box$). We here simply report our progress on this question. In unpublished work, we have shown a series of inconsistency results for extensions of **KD** and one consistency result, for the logic **KDDc**, where (Dc) is the axiom schema $\Diamond\varphi \to \Box\varphi$.

7.3 Gödel-McKinsey-Tarski Translation of Naive comprehension

Our work in this paper has been guided by an interest in the simplest modal comprehension principles. More complex modal principles may nevertheless have simple motivations. The Gödel-McKinsey-Tarski translation of naive comprehension is not our principle but the following:

$$\Box\Diamond(\exists y)\Box(\forall x)\Box(\Box x \in y \leftrightarrow \Box\varphi). \qquad \text{(CompGMT)}$$

That principle is inconsistent in **S4** by the inconsistency of naive comprehension in intuitionistic logic. In fact, we can strengthen this result, by simply considering the Gödel-McKinsey-Tarski translation of the Russell set:

$$\Box\Diamond(\exists y)\Box(\forall x)\Box(\Box x \in y \leftrightarrow \Box\neg\Box\varphi). \qquad \text{(RussellGMT)}$$

Proposition 7.3. (RussellGMT) *is inconsistent in* **T**.

Proof.

$$
\begin{array}{lll}
(1) & \forall x\Box(\Box x \in R \leftrightarrow \Box\neg\Box x \in x) \rightarrow (\Box R \in R \leftrightarrow \Box\neg\Box R \in R) & \forall\text{-out} \\
(2) & \forall x\Box(\Box x \in R \leftrightarrow \Box\neg\Box x \in x) \rightarrow \neg\Box R \in R & \text{1, T, PL} \\
(3) & \Box\forall x\Box(\Box x \in R \leftrightarrow \Box\neg\Box x \in x) \rightarrow \Box\neg\Box R \in R & \text{2, RM} \\
(4) & \Box\forall x\Box(\Box x \in R \leftrightarrow \Box\neg\Box x \in x) \rightarrow \Box R \in R & \text{1, 3, PL} \\
(5) & \neg\Box\forall x\Box(\Box x \in R \leftrightarrow \Box\neg\Box x \in x) & \text{2, 4, PL} \\
(6) & \Box\Box\forall y\neg\Box\forall x\Box(\Box x \in y \leftrightarrow \Box\neg\Box x \in x) & \text{RN, } \forall\text{-intro}
\end{array}
$$

QED

This result raises a third question which we will leave for future work. If we consider the non-modal propositional logic generated by taking the theorems of some modal system Σ which are also in the image of the Gödel-McKinsey-Tarski translation, then the consistency of $\Sigma+$ (CompGMT) would imply the consistency of naive comprehension in the logic generated as described. Perhaps modal model constructions such as the one in the previous section could be used to show the consistency of naive comprehension in sub-intuitionistic logics. More generally the study of sub-intuitionistic logics generated in this way, and the correspondence between modal axioms of the translation and non-modal axioms of the pre-translated logic seems an interesting issue which we will also leave for future work.

For now, we close by recalling that the axiomatic set theory obtained by adding (Comp$\Box$) to **S5** is too weak to form the foundation for mathematics. ($\Box$Comp$\Box$) was certainly more promising, but as we have seen at length, it is inconsistent in **T**, while all of its close cousins are also inconsistent in **S4** (and hence **S5**). At least for the modal logics considered in this paper, the answer to the question of our title is definitively "no".

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