Differential algebraic equations from definability

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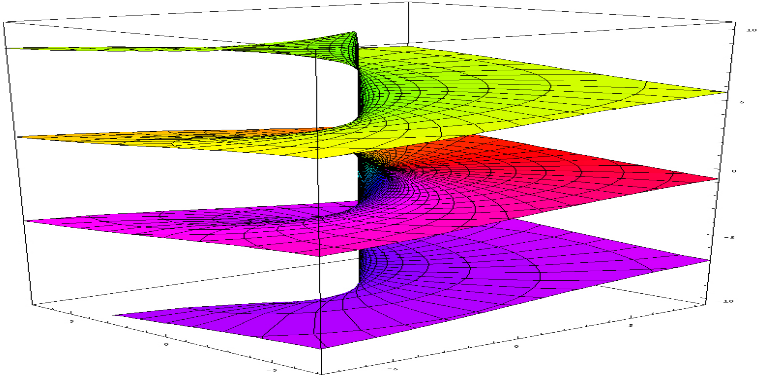
Is the logarithm a function?



The exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ has a many-valued analytic inverse $\log : \mathbb{C}^\times \to \mathbb{C}$ where log is well-defined only up to the adding an element of $2\pi i\mathbb{Z}$.

The logarithmic derivative

Treating exp and log as functions on functions does not help: If U is some connected Riemann surface and $f : U \to \mathbb{C}^\times$ is analytic, then we deduce a "function" $\log(f) : U \to \mathbb{C}$.

However, because $\log(f)$ is well-defined up to an additive constant, $\partial \log(f) := \frac{d}{dz}(\log(f))$ is a well defined function. That is, for $M = \mathscr{M}(U)$ the differential field of meromorphic functions on U we have a well-defined differential-analytic function $\partial \log : M^\times \to M$.

Of course, one computes that $\partial \log(f) = \frac{f'}{f}$ is, in fact, differential algebraic.

Explanation?

Why is the logarithmic differential algebraic?

- What a silly question! The logarithm is the very first transcendental function whose derivative is computed in a standard calculus course. The differential algebraicity of $\frac{d}{dz}(\log(f))$ is merely a consequence of an elementary calculation.
- The usual logarithmic derivative is an instance of Kolchin's general theory of algebraic logarithmic derivatives on algebraic groups in which the differential algebraicity is explained by the triviality of the tangent bundle of an algebraic group.
- It can also be seen as an instance of the main theorem to be discussed today: certain kinds of differential analytic functions constructed from covering maps are automatically differential algebraic due to two key ideas from logic: elimination of imaginaries and the Peterzil-Starchenko theory of o-minimal complex analysis.

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UNE THÉORIE DE GALOIS IMAGINAIRE

BRUNO POIZAT

Introduction. La communauté mathématique doit être reconnaissante à Saharon Shelah pour une invention d'une ingénieuse simplicité, celle d'avoir associé à chaque structure M une structure M^{eq} comprenant, outre les éléments de M, des "éléments imaginaires" qui sont virtuellement présents dans M. La finalité de cette construction est de pourvoir toute formule $f(\bar{x}, \bar{a})$ à paramètres dans M, et même dans M^{eq}, d'un ensemble de définition minimum; tout cela est rappelé dans la première section du présent article.

On peut a priori douter de l'utilité d'une construction si innocente, dont la propriété fondamentale est pratiquement évidente; et pourtant elle a été abondamment montrée par son auteur, à qui elle a permis, dans les théorèmes de classification des modèles, une décomposition des types en éléments simples, qu'on ne voit pas dans M.

Cette construction d'un plus petit ensemble de définition pour une formule en rappelle une autre, qui est bien connue des algébristes, celle du corps de définition d'un idéal. Et comme la théorie des corps algébriquement clos de caractéristique donnée élimine les quanteurs, on a le sentiment que l'adjonction d'imaginaires aux modèles de cette théorie est inutile, en un mot qu'elle "élimine les imaginaires"; pour vérifier le bien-fondé de ce sentiment, il convient d'abord de préciser ce qu'on entend par là, ce qui est fait dans la deuxième section.

Imaginaries, a definition

If M is an $\mathscr{L}$-structure and $E \subseteq M^n \times M^n$ is an $\mathscr{L}$-definable equivalence relation on some Cartesian power of M, then each E-equivalence class in M^n/E is called an imaginary.

Shelah associates to any structure M, a new multisorted structure M^{eq}, whose elements are exactly the imaginaries of M.

The point of Poizat's notion of elimination of imaginaries is that for some structures these imaginary elements which are always virtually present in M are really there.

Elimination of imaginaries

We say that the theory T eliminates imaginaries if for any model $M \models T$ and any definable equivalence relation $E \subseteq M^n \times M^n$ on some Cartesian power of M, there is a definable function $f : M^n \to M^m$ for which for all $a, b \in M^n$ one has

$$f(a) = f(b) \iff aEb \ .$$

That is, each imaginary element $[a]_E$ is interdefinable with a finite sequence of real elements $f(a) \in M^m$.

Examples

- The theory of equality does not eliminate imaginaries. For example, the equivalence relation E on pairs defined by $\langle u, v \rangle E \langle x, y \rangle :\Longleftrightarrow (u = x \;\&\; v = y) \vee (u = y \;\&\; v = x)$ is not eliminable.
- Peano Arithmetic eliminates imaginaries
- ZFC eliminates imaginaries
- The theory of algebraically closed fields eliminates imaginaries.
- The theory of differentially closed fields eliminates imaginaries.

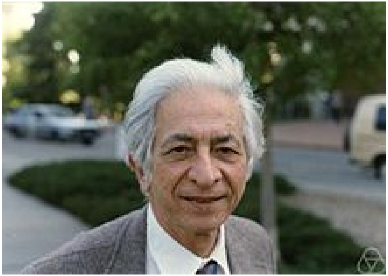
Differentially closed fields

The theory DCF_0 of differentially closed fields of characteristic zero is the model companion of the theory of differential fields of characteristic zero.

Here, a differential field is a field K given together with a derivation $\partial : K \to K$, an additive map ($\partial(x + y) = \partial(x) + \partial(y)$) satisfying the Leibniz rule ($\partial(xy) = x\partial(y) + \partial(x)y$).

Models of DCF_0

Natural models of DCF_0 are not so easy to describe, but it follows from a theorem of Seidenberg that every countable differential field of characteristic zero may be realized as a subdifferential field of germs of meromorphic functions.



Eliminating an imaginary in DCF_0

In a differential field (K, ∂) we say that $c \in K$ is constant if $\partial(c) = 0$. Write $C = C(K)$ for the set (field) of all constants in K.

Define an equivalence relation

$$f \sim g :\Longleftrightarrow (\exists c \text{ constant }) \;\; f = g + c \; .$$

Then $f \sim g \Longleftrightarrow \partial(f) = \partial(g)$. That is, the function $\partial : K \to K$ eliminates the equivalence relation $\sim$.

Fractional linear equivalence

For f and g nonconstant, define

$$f \sim g :\Longleftrightarrow (\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C))\ f = \frac{ag+b}{cg+d}\ .$$

Since $\sim$ is a definable equivalence relation, there should be a definable function whose fibres are precisely the $\sim$-equivalence classes.

Schwarzian derivarive

Define the *Schwarzian derivative* by

$$S(x) := (\frac{x''}{x'})' - \frac{1}{2}(\frac{x''}{x'})^2 \ .$$

where we write x' for the derivative $\partial(x)$.
Then $f \sim g \iff S(f) = S(g)$.

Generalized Schwarzians

Let (K, ∂) be a differential field of characteristic zero with algebraically closed field of constants C, X an algebraic variety over C and $G \curvearrowright X$ is an algebraic group acting on X, then there is a piecewise differential rational function η defined on $X(K)$ so that for $a, b \in X(K)$ one has

$$\eta(a) = \eta(b) :\Longleftrightarrow (\exists \gamma \in G(C))\ \gamma \cdot a = b\ .$$

Here a differential rational function is one whose coordinates take the form

$$(x_1, \ldots, x_n) \mapsto \frac{P(x_1, \ldots, x_n; x_1', \ldots, x_n'; \ldots; x_1^{(m)}, \ldots, x_n^{(m)})}{Q(x_1, \ldots, x_n; x_1', \ldots, x_n'; \ldots; x_1^{(m)}, \ldots, x_n^{(m)})}$$

for some polynomials P and Q.

The general logarithmic derivative problem, set-up

We are given:

- complex algebraic groups $K < G$,
- a complex submanifold $D \subseteq (G/K)(\mathbb{C})$,
- a discrete subgroup $\Gamma < G(\mathbb{C})$ for which $\Gamma \curvearrowright D$,
- an algebraic variety X, and
- an analytic covering map $\pi : D \to X(\mathbb{C})$ expressing $X(\mathbb{C}) = \Gamma \backslash D$.

For example, we may take $K = \{0\}$, $G = (\mathbb{C}, +) = D$ with G acting on itself by translation and $\Gamma = 2\pi i \mathbb{Z}$, then $\pi : D \to X(\mathbb{C}) := \mathbb{C}^\times$ is simply the exponential map.

Generalized differential logarithm

As with the logarithm, the inverse function $\pi^{-1} : X(\mathbb{C}) \to D$ is locally analytic, but is only well-defined up to the action of Γ and in the same way if U is some connected Riemann surface and $f : U \to X(\mathbb{C})$ is analytic, then we deduce a mutlivalued function $\pi^{-1}(f)$. Put another way, if $M = \mathscr{M}(U)$ is the differential field of meromorphic functions on U, we have a multivalued analytic function $\pi^{-1} : X(M) \to (G/K)(M)$ well-defined up to the action of Γ.

Applying our generalized Schwartzian η corresponding to the action of $G(\mathbb{C})$, we have a well-defined differential analytic function χ defined by $\chi := \eta \circ (\pi^{-1})$.

GEOMÉTRIE ALGÉBRIQUE ET GÉOMÉTRIE ANALYTIQUE

par Jean-Pierre SERRE.

INTRODUCTION

Soit X une variété algébrique projective, définie sur le corps des nombres complexes. L'étude de X peut être entreprise de deux points de vue : le point de vue *algébrique*, dans lequel on s'intéresse aux anneaux locaux des points de X, aux applications rationnelles, ou régulières, de X dans d'autres variétés, et le point de vue *analytique* (parfois appelé « transcendant ») dans lequel c'est la notion de fonction holomorphe sur X qui joue le principal rôle. On sait que ce second point de vue s'est révélé particulièrement fécond lorsque X est non singulière, cette hypothèse permettant de lui appliquer toutes les ressources de la théorie des variétés kählériennes (formes harmoniques, courants, cobordisme, etc.)

Dans de nombreuses questions, les deux points de vue conduisent à des résultats essentiellement équivalents, bien que par des méthodes très différentes. Par exemple, on sait que les formes différentielles holomorphes en tout point de X ne sont pas autre chose que les formes différentielles rationnelles qui sont partout « de première espèce » (la variété X étant encore supposée non singulière); le théorème de Chow, d'après lequel tout sous-espace analytique fermé de X est une variété algébrique, est un autre exemple du même type.

Le but principal du présent mémoire est d'étendre cette équivalence aux *faisceaux cohérents*; de façon précise, nous

Peterzil-Starchenko GAGA

Theorem If X is a complex algebraic variety and $Y \subseteq X(\mathbb{C})$ is an o-minimally definable, analytically constructible set, then Y is algebraically constructible.

O-minimal complex analysis

O-minimality is a property of ordered structures: $(R, <, \ldots)$ is o-minimal if every definable subset of R is a finite union of points and intervals.

Just as $\mathbb{C}$ may be interpreted in $\mathbb{R}$ as $\mathbb{R}^2$ with addition and multiplication defined via polynomial functions, likewise if $\mathscr{R} := (R, <, +, \cdot, \ldots)$ is an o-minimal expansion of an ordered field, then $C := R[i]$ may be interpreted in R. A definable function $f : U \to C$ (where $U \subseteq C$ is an open subset) is $\mathscr{R}$-analytic if it is $\mathscr{R}$-definable and for every $a \in U$ the derivative $f'(a) := \lim_{C \ni \epsilon \to 0} \frac{f(a+\epsilon)-f(a)}{\epsilon}$ exists.

Peterzil and Starchenko develop $\mathscr{R}$-complex analysis without the benefit of some standard tools in complex analysis (Taylor series developments, analytic continuation, integration over curves, *et cetera*).

Algebraicity of generalized differential logarithms

We are given:

- complex algebraic groups $K < G$,
- a complex submanifold $D \subseteq (G/K)(\mathbb{C})$,
- a discrete, Zariski dense subgroup $\Gamma < G(\mathbb{C})$ for which $\Gamma \curvearrowright D$,
- an algebraic variety X, and
- an analytic covering map $\pi : D \to X(\mathbb{C})$ expressing $X(\mathbb{C}) = \Gamma \backslash D$.

Assume moreover that there is a set $F \subseteq D$ which is definable in an o-minimal expansion of $(\mathbb{R}, <, +, \cdot, 0, 1)$ for which $\pi \upharpoonright F : F \to X(\mathbb{C})$ is definable and surjective.

Then the differential analytic map $\chi := \eta \circ \pi^{-1}$ is in fact differentially algebaically constructible where η is the differentially constructible function expressing $G(\mathbb{C}) \backslash (G/K)(\mathbb{U})$ as a definable set.

When does the theorem apply?

The standard o-minimal structure for these purposes is $\mathbb{R}_{\mathrm{an,exp}}$, in which one is allowed all polynomials over the reals, the real exponential function, and real analytic functions restricted to compact boxes (and any other function built from these).

- $\exp_A : \mathbb{C}^g \to A(\mathbb{C})$ where A is an abelian variety of dimension g
- $j : \mathfrak{h} \to \mathbb{A}^1(\mathbb{C})$, the analytic j-function expressing $\mathbb{A}^1 = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$
- More generally, theta functions and covering maps associated to moduli spaces of abelian varieties and for their universal families.
- Manin homomorphisms
- universal covering maps of hyperbolic curves

Conclusions

- In this lecture, I have focused on the ideas from logic used in the construction of these algebraic differential operators. For details on the proofs and the applications to Picard-Fuchs equations, Manin homomorphisms, automorphic functions, *et cetera* see my paper "Algebraic differential equations from covering maps", `arXiv:1408.5177`.
- The Peterzil-Starchenko GAGA theorem is very strong, but to my knowledge it has been applied in only one other paper, of Pila and Tsimerman on functional transcendence. What more can it say about the relation between algebraic and tame analytic geometry?
- Explicit forms of the generalized Schwarzians and generalized logarithmic derivatives would make them more useful for computations. In principle, the generalized Schwarzian may be computed using elimination theory. Can we compute an algebraic representation of the generalized logarithmic derivatives?