Transferring Imaginaries

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- Let $(X_y)_{y \in Y}$ be an $\varnothing$-definable family of sets. Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$. The set $Y/{\equiv}$ is a "moduli space" for the family $(X_y)_{y \in Y}$.

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- Let G be a definable group and $H \trianglelefteq G$ be a definable subgroup. The group G/H is interpretable but *a priori* not definable.

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Definition

A theory T eliminates imaginaries if for all $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f defined on D such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$

What is your quest?

Proposition (Shelah, 1978)

Let $A \subseteq M \vDash T$ stable, $p \in \mathcal{S}(A)$ and $p_1, p_2 \in \mathcal{S}(M)$ be two distinct extensions of p to M definable over A. Then there exists an $\mathcal{L}(A)$-definable finite equivalence relation E and $a_1, a_2 \in M$ such that:

- a_1 and a_2 are not E-equivalent;
- $p_i(x) \vdash xEa_i$.

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Definition

- A type p over M is said to be definable (over A) if for all formula $\phi(x; y)$ there is a formula $\theta(y)$ such that

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- A theory is said to be stable if every type over every model of T is definable.

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- Proving elimination of imaginaries in specific structures can have (more or less direct) applications.

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$$\mathcal{L}^{\text{eq}} := \mathcal{L} \cup \{S_E, f_E \mid E \text{ is an } \varnothing\text{-definable equivalence relation}\}$$

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Proposition

A theory T (with two constants) eliminates imaginaries if and only if for all $M \models T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $a \in M$ such that

$$e \in \mathrm{dcl}^{\mathrm{eq}}(a) \text{ and } a \in \mathrm{dcl}^{\mathrm{eq}}(e).$$

Weak elimination

Definition

A theory T weakly eliminates imaginaries if for all $M \vDash T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $a \in M$ such that

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- Any strongly minimal theory weakly eliminates imaginaries.

Encoding functions

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2. Every set definable in models of T has a smallest (algebraically closed) set of definition.
3. Every finite valued function $M \to M$ definable in $M \vDash T$ has a smallest (algebraically closed) set of definition.

Covering functions

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    - Unclear how to recover f from g.

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Let T' be a theory of fields such that, for all $M \models T'$ and $A \subseteq M$:

1. $\mathrm{dcl}(A) = \mathrm{acl}(A) \subseteq \overline{A}^{\mathrm{alg}}$;
2. Every definable $X \subseteq M$ has a smallest subset of definition.

Then T eliminates imaginaries.

Remark

Hypothesis 1 holds in $\mathbb{Q}_p$ but not hypothesis 2 (in the language of rings).

Covering functions

Proposition (Hrushovski-Martin-R., 2014)

Let T be an $\mathcal{L}$-theory that eliminates quantifiers and imaginaries and $T' \supseteq T_\forall$ an $\mathcal{L}'$-theory.

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$$\sigma(e) = e \text{ if and only if } \sigma(e') = e';$$

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4. Assume $A = \mathrm{acl}_{\mathcal{L}'}(A)$ and let $p \in \mathcal{S}_1^{\mathcal{L}'}(A)$. Then there exists $\widetilde{p} \in \mathcal{S}_1^{\mathcal{L}}(M)$ definable over A such that $p \cup \widetilde{p}|_{M'}$ is consistent.

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Covering functions

Proposition

Let T_i be an $\mathcal{L}_i$-theory that eliminates quantifiers and imaginaries and $T' \supseteq \bigcup_i T_{i,\forall}$ an $\mathcal{L}'$-theory. Assume that, for all $M' \vDash T'$, $M_i \vDash T_i$ containing M' and $A \subseteq M'$:

1. $\mathrm{dcl}_{\mathcal{L}'}(A) = \mathrm{acl}_{\mathcal{L}'}(A) \subseteq \mathrm{acl}_{\mathcal{L}_i}(A)$;
2. Every definable $X \subseteq M'$ has a smallest subset of definition;
3. For all $e \in \mathrm{dcl}_{M_i}(M')$, there exists $e' \in M'$ such that for all $\sigma \in \mathrm{Aut}(M_i)$ stabilising M' globally,
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4. Assume $A = \mathrm{acl}_{\mathcal{L}'}(A)$ and let $p \in \mathcal{S}_1^{\mathcal{L}'}(A)$. Then there exists $\widetilde{p}_i \in \mathcal{S}_1^{\mathcal{L}_i}(M_i)$ definable over A such that $p \cup \bigcup_i \widetilde{p}_i|_{M'}$ is consistent.

Then T' weakly eliminates imaginaries.

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- All the imaginaries in $\prod_p \mathbb{Q}_p / \mathfrak{U}$ come from ACVF.

Adding new functions

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Proposition (Chatzidakis-Pillay, 1998)

Assume T is strongly minimal, then T_A weakly eliminates imaginaries.

Proposition (Hrushovski, 2012)

Let T be a stable theory that eliminates imaginaries. Assume that T has 3-uniqueness, then T_A eliminates imaginaries.

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$$\begin{array}{cccc} \nabla_\omega : & \mathcal{S}_x^{\mathcal{L}'}(M) & \to & \mathcal{S}_{x_\omega}^{\mathcal{L}}(M) \\ & \mathrm{tp}_{\mathcal{L}'}(a/M) & \mapsto & \mathrm{tp}_{\mathcal{L}}(f_\omega(a)/M) \end{array}$$

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- We assume that ∇_ω is injective (this is a form of quantifier elimination).
- That does not, in general, hold in T_A.
- It does hold in differentially closed fields of characteristic zero and separably closed fields of finite imperfection degree (and their valued equivalents).

Imaginaries and definable types

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Let T be a theory such that:

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Remark

It suffices to prove hypothesis 1 in dimension 1.

Prolongations and canonical basis

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In the case of differentially closed fields $(K, 0, 1, +, -, \cdot, \delta)$:

- Hypothesis I is true because DCF_0 is stable.
- Let $M \models \mathrm{DCF}_0$ and $p \in \mathcal{S}^{\mathcal{L}_\partial}(M)$.
- Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$ and assume p is $\mathcal{L}_\partial^{\mathrm{eq}}(A)$-definable. By elimination of imaginaries in ACF, the canonical basis of $\nabla_\omega(p)$ is contained in $\mathbf{K}(A)$. In particular, p is $\mathcal{L}_\partial(\mathbf{K}(A))$-definable.

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- If p is $\mathcal{L}'(M)$-definable, there is no reason for $\nabla_\omega(p)$ to be $\mathcal{L}(M)$-definable.
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and we wish this set to be $\mathcal{L}$-definable.

NIP theories

Definition

Let $\phi(x;y)$ be a formula and M a structure, we say that ϕ has the independence property in M if there exists $(a_n)_{n\in\mathbb{N}}$ and $(b_X)_{X\subseteq\mathbb{N}}$ such that:

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Definition (Stable embeddedness)

Let M be some structure and $A \subseteq M$. We say that A is stably embedded in M if for all formula $\phi(x; y)$ and all $c \in M$, there exists a formula $\psi(x; z)$ such that

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for some tuple $a \in A$.

Definition (Uniform stable embeddedness)

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Let T be an NIP be an $\mathcal{L}$-theory and $\widetilde{T}$ be a complete enrichment of T in a language $\widetilde{\mathcal{L}}$. Assume that there exits $M \vDash \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.

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In particular, any $\mathcal{L}$-type which is $\widetilde{\mathcal{L}}$-definable is in fact $\mathcal{L}$-definable.

Prolongations and canonical basis II

Proposition

Let T be some $\mathcal{L}$-theory that eliminates imaginaries, f be new function symbol and $T' \supseteq T$ be a complete $\mathcal{L} \cup \{f\}$-theory. Assume that:

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Thanks!