Transferring Imaginaries

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November 6th 2015
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**Example**

- Let $(X_y)_{y \in Y}$ be an $\emptyset$-definable family of sets. Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$. The set $Y/\equiv$ is a “moduli space” for the family $(X_y)_{y \in Y}$.
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- Let $G$ be a definable group and $H \trianglelefteq G$ be a definable subgroup. The group $G/H$ is interpretable but *a priori* not definable.
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- Let $G$ be a definable group and $H \triangleleft G$ be a definable subgroup. The group $G/H$ is interpretable but *a priori* not definable.

**Definition**

A theory $T$ eliminates imaginaries if for all $\emptyset$-definable equivalence relation $E \subseteq D^2$, there exists an $\emptyset$-definable function $f$ defined on $D$ such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$
What is your quest?

Proposition (Shelah, 1978)

Let $A \subseteq M \models T$ be stable, $p \in S(A)$ and $p_1, p_2 \in S(M)$ be two distinct extensions of $p$ to $M$ definable over $A$. Then there exists an $\mathcal{L}(A)$-definable finite equivalence relation $E$ and $a_1, a_2 \in M$ such that:

- $a_1$ and $a_2$ are not $E$-equivalent;
- $p_i(x) \vdash x E a_i$. 
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**Definition**

- A type $p$ over $M$ is said to be definable (over $A$) if for all formula $\phi(x; y)$ there is a formula $\theta(y)$ such that

  $$\phi(x; a) \in p \text{ if and only if } M \models \theta(a).$$

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  \[ \phi(x; a) \in p \text{ if and only if } M \models \theta(a). \]
  We will often write $d_p x \phi(x; y) = \theta(y)$.
- A theory is said to be stable if every type over every model of $T$ is definable.

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- If $p$ is a definable type, then $p$ has a smallest (definably closed) set of definition. It is called the canonical basis of $p$. 
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- Proving elimination of imaginaries in specific structures can have (more or less direct) applications.
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- $O$-minimal groups eliminate imaginaries.
  For example, any $O$-minimal enrichment of $(\mathbb{R}, 0, 1, +, −, \cdot)$. 

- $\mathbb{Q}_p$ does not eliminate imaginaries in the ring language:
  - $\mathbb{Z}$ can be interpreted as $\mathbb{Q}^{\star}p / \mathbb{Z}^{\star}p$;
  - All infinite definable subsets of $\mathbb{Q}$ have cardinality continuum.

- Henselian valued fields do not eliminate imaginaries in the language of valued rings.
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Shelah’s eq construction

Let $T$ be a theory. For all $\emptyset$-definable equivalence relation $E \subseteq \prod_{i} S_i$, let $S_E$ be a new sort and $f_E : \prod S_i \to S_E$ be a new function symbol. Let $L_{eq} : = L \cup \{ S_E; f_E \mid E$ is an $\emptyset$-definable equivalence relation $\}$ and $T_{eq} : = T \cup \{ f_E$ is onto and $\forall x; y ( f_E(x) = f_E(y) \leftrightarrow x E y) \}$.
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**Remark**

- Let $M \models T$, then $M$ can naturally be enriched into a model of $T^{eq}$ that we denote $M^{eq}$. 
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**Remark**

- Let $M \models T$, then $M$ can naturally be enriched into a model of $T^{eq}$ that we denote $M^{eq}$.
- We will denote by $\mathcal{R}$ the set of $\mathcal{L}$-sorts. They are called the real sorts.
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- We will denote by $dcl^{eq}$ ($acl^{eq}$) the definable (algebraic) closure in $T^{eq}$.
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Proposition

A theory $T$ (with two constants) eliminates imaginaries if and only if for all $M \models T$ and $e \in M_{eq}$, there exists a tuple $a \in M$ such that

$$e \in \text{dcl}_{eq}(a) \text{ and } a \in \text{dcl}_{eq}(e).$$
Weak elimination

Definition

A theory $T$ weakly eliminates imaginaries if for all $M \models T$ and $e \in M^{eq}$, there exists a tuple $a \in M$ such that

$$e \in dcl^{eq}(a) \text{ and } a \in acl^{eq}(e).$$
## Weak elimination

### Definition

A theory $T$ weakly eliminates imaginaries if for all $M \models T$ and $e \in M^\text{eq}$, there exists a tuple $a \in M$ such that

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### Proposition

A theory $T$ eliminates imaginaries if and only if:

1. $T$ weakly eliminates imaginaries.
2. For all $M \models T$, the quotient of $M^n$ by the action of $S_n$ is represented.
Weak elimination

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- Infinite sets weakly eliminate imaginaries.
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**Example**
- Infinite sets weakly eliminate imaginaries.
- Any strongly minimal theory weakly eliminates imaginaries.
A finite valued function $X \to Y$ is a subset of $X \times Y$ such that for all $x \in X$, the set $Y_x$ is finite.
A finite valued function \( X \rightarrow Y \) is a subset of \( X \times Y \) such that for all \( x \in X \), the set \( Y_x \) is finite.

### Proposition

The following are equivalent:

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The following are equivalent:

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2. Every set definable in models of $T$ has a smallest (algebraically closed) set of definition.
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**Proposition**

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1. $T$ weakly eliminates imaginaries
2. Every set definable in models of $T$ has a smallest (algebraically closed) set of definition.
3. Every finite valued function $M \to M$ definable in $M \models T$ has a smallest (algebraically closed) set of definition.
Covering functions

Let $T$ be an $L$-theory and $T' \supseteq T$ be an $L'$-theory. Let $M' \models T'$ and $M \models T$ containing $M'$.

Assume that every finite valued function $f$ definable in $M'$ is covered by a finite valued function $g$ defined in $M$.

One would like to deduce elimination of imaginaries in $T'$ from elimination of imaginaries in $T$.

There are a number of problems:

- No control the domain of $f$.
- $g$ is not canonical (unless it can somehow be taken minimal).
- The smallest set of definition of $g$ might contain points from $M \setminus M'$.
- Unclear how to recover $f$ from $g$.
Let $T$ be an $\mathcal{L}$-theory and $T' \supseteq T_\forall$ be an $\mathcal{L}'$-theory. Let $M' \models T'$ and $M \models T$ containing $M'$. 

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In the case of the field \((\mathbb{R}, 0, 1, +, -, \cdot)\):
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- Take any finite valued function \(f\) definable in \(\mathbb{R}\). Let \(g\) be the Zariski closure of \(f\). Then \(g\) is a finite valued function definable in \(\mathbb{C}\).
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- Let $A \subseteq \mathbb{C}$ be the the smallest set of definition of $g$.
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- Let \(A \subseteq \mathbb{C}\) be the the smallest set of definition of \(g\).
- The smallest set of definition of \(g \cap \mathbb{R}\) is \(A \cap \mathbb{R}\).
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- Let \(A \subseteq \mathbb{C}\) be the smallest set of definition of \(g\).
- The smallest set of definition of \(g \cap \mathbb{R}\) is \(A \cap \mathbb{R}\).
- \(f\) can be recovered from \(g \cap \mathbb{R}\) using the order and the fact that every definable \(X \subseteq \mathbb{R}\) has a smallest subset of definition.
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**Proposition (Hrushovski-Martin-R., 2014)**

Let \(T'\) be a theory of fields such that, for all \(M \models T'\) and \(A \subseteq M\):

Then \(T\) eliminates imaginaries.
Covering functions

In the case of the field \((\mathbb{R}, 0, 1, +, -, \cdot)\):

- Take any finite valued function \(f\) definable in \(\mathbb{R}\). Let \(g\) be the Zariski closure of \(f\). Then \(g\) is a finite valued function definable in \(\mathbb{C}\).
- Let \(A \subseteq \mathbb{C}\) be the the smallest set of definition of \(g\).
- The smallest set of definition of \(g \cap \mathbb{R}\) is \(A \cap \mathbb{R}\).
- \(f\) can be recovered from \(g \cap \mathbb{R}\) using the order and the fact that every definable \(X \subseteq \mathbb{R}\) has a smallest subset of definition.

**Proposition (Hrushovski-Martin-R., 2014)**

Let \(T'\) be a theory of fields such that, for all \(M \models T'\) and \(A \subseteq M\):

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**Remark**

Hypothesis 1 holds in \(\mathbb{Q}_p\) but not hypothesis 2 (in the language of rings).
Covering functions

Proposition (Hrushovski-Martin-R., 2014)

Let $T$ be an $\mathcal{L}$-theory that eliminates quantifiers and imaginaries and $T' \supseteq T_\forall$ an $\mathcal{L}'$-theory.

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Let $T$ be an $\mathcal{L}$-theory that eliminates quantifiers and imaginaries and $T' \supseteq T_\forall$ an $\mathcal{L}'$-theory. Assume that, for all $M' \models T'$, $M \models T$ containing $M'$ and $A \subseteq M'$:

1. $dcl_{\mathcal{L}'}(A) = acl_{\mathcal{L}'}(A) \subseteq acl_{\mathcal{L}}(A)$;
2. Every definable $X \subseteq M'$ has a smallest subset of definition;
3. For all $e \in dcl_{\mathcal{M}}(M')$, there exists $e' \in M'$ such that for all $s \in \text{Aut}(M')$ stabilising $M'$ globally, $(e) = e$ if and only if $(e') = e'$;
4. Assume $A = acl_{\mathcal{L}'}(A)$ and let $p \in S_{\mathcal{L}'}^1(A)$. Then there exists $\tilde{p} \in S_{\mathcal{L}'}^1(M)$ definable over $A$ such that $p \cup \tilde{p} \mid M'$ is consistent.

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Then $T'$ eliminates imaginaries.
Proposition

Let $T_i$ be an $\mathcal{L}_i$-theory that eliminates quantifiers and imaginaries and $T' \supseteq \bigcup_i T_i \forall$ an $\mathcal{L}'$-theory. Assume that, for all $M' \models T'$, $M_i \models T_i$ containing $M'$ and $A \subseteq M'$:

1. $dcl_{\mathcal{L}'}(A) = acl_{\mathcal{L}'}(A) \subseteq acl_{\mathcal{L}_i}(A)$;
2. Every definable $X \subseteq M'$ has a smallest subset of definition;
3. For all $e \in dcl_{M_i}(M')$, there exists $e' \in M'$ such that for all $\sigma \in \text{Aut}(M_i)$ stabilising $M'$ globally,
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Then $T'$ weakly eliminates imaginaries.
Some results

▸ All the imaginaries in $\mathbb{R}$ come from ACF (and hence they can be eliminated).

▸ All the imaginaries in real closed valued fields come from ACVF (whose imaginaries were described by Haskell, Hrushovski and Macpherson).

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Adding new functions

If $T$ is an $L$-theory, we may want to form $T \cup \{\}$ -theory of models of $T$ with an automorphism.

We will mainly be interested in $T_A$, the model companion of $T$, if it exists (and from now on, we will assume it exists).

**Proposition (Chatzidakis-Pillay, 1998)**
Assume $T$ is strongly minimal, then $T_A$ weakly eliminates imaginaries.

**Proposition (Hrushovski, 2012)**
Let $T$ be a stable theory that eliminates imaginaries. Assume that $T$ has $3$-uniqueness, then $T_A$ eliminates imaginaries.
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Adding new functions

- Let $T$ be some $\mathcal{L}$-theory, $f$ be new function symbol and $T' \supseteq T$ be an $\mathcal{L} \cup \{f\}$-theory.
- Let $M \models T'$. We define:

$$\nabla_\omega : \quad S^\mathcal{L}_x (M) \quad \rightarrow \quad S^\mathcal{L}_{x,\omega} (M)$$

$$\text{tp}_{\mathcal{L}'} (a/M) \quad \mapsto \quad \text{tp}_\mathcal{L} (f_\omega (a)/M)$$

where $f_\omega (a) = (f^n (a))_{n \in \mathbb{N}}$. 

- We assume that $\nabla_\omega$ is injective (this is a form of quantifier elimination).
- That does not, in general, hold in $T_A$.
- It does hold in differentially closed fields of characteristic zero and separably closed fields of finite imperfection degree (and their valued equivalents).
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Proposition (Hrushovski, 2014)

Let $T$ be a theory such that:

1. For every definable set $X$ there exists an $L_{eq}(acl_{eq}(⌜X⌝))$-definable type $p$ which is consistent with $X$.
2. Let $A = acl_{eq}(A) \subseteq M_{eq} \models T_{eq}$. If $p \in S(M)$ is $L_{eq}(A)$-definable, then $p$ is $L(R(A))$-definable.

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Remark

It suffices to prove hypothesis 1 in dimension 1.
In the case of differentially closed fields \((K, 0, 1, +, -, \cdot, \delta)\):
Prolongations and canonical basis

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In the case of differentially closed fields \((K, 0, 1, +, -, \cdot, \delta)\):

- Hypothesis 1 is true because \(DCF_0\) is stable.
- Let \(M \models DCF_0\) and \(p \in S^{\mathcal{L}\delta}(M)\).
- Let \(A = acl^eq(A) \subseteq M^eq\) and assume \(p\) is \(\mathcal{L}_\delta^eq(A)\)-definable. By elimination of imaginaries in ACF, the canonical basis of \(\nabla\omega(p)\) is contained in \(K(A)\). In particular, \(p\) is \(\mathcal{L}_\delta(K(A))\)-definable.
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and we wish this set to be $\mathcal{L}$-definable.
NIP theories

Definition

Let $\phi(x;y)$ be a formula and $M$ a structure, we say that $\phi$ has the independence property in $M$ if there exists $(a_n)_{n \in \mathbb{N}}$ and $(b_X)_{X \subseteq \mathbb{N}}$ such that:

$$M \models \phi(a_n; b_X) \text{ if and only if } n \in X$$

We say that the theory $T$ is NIP (not the independence property) if no formula has the independence property in any model of $T$. 

Example

- All stable theories are NIP.
- All $O$-minimal theories are NIP.
- ACVF is NIP.
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Definable types in enrichments of NIP theories

**Definition (Stable embeddedness)**

Let $M$ be some structure and $A \subseteq M$. We say that $A$ is stably embedded in $M$ if for all formula $\phi(x;y)$ and all $c \in M$, there exists a formula $\psi(x;z)$ such that

$$\phi(A; c) = \psi(A; a)$$

for some tuple $a \in A$. 

**Proposition (Simon-R., 2015)**

Let $T$ be an NIP $L$-theory and $\bar{T}$ be a complete enrichment of $T$ in a language $\bar{L}$. Assume that there exists $M \models \bar{T}$ such that $M \models L$ is uniformly stably embedded in every elementary extension. Let $X$ be a set that is both externally $L$-definable and $\bar{L}$-definable, then $X$ is $L$-definable.

In particular, any $L$-type which is $\bar{L}$-definable is in fact $L$-definable.
Definable types in enrichments of NIP theories

Definition (Uniform stable embeddedness)

Let $M$ be some structure and $A \subseteq M$. We say that $A$ is uniformly stably embedded in $M$ if for all formula $\phi(x; y)$, there exists a formula $\psi(x; z)$ such that for all tuple $c \in M$,

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Proposition (Simon-R., 2015)

Let $T$ be an NIP be an $L$-theory and $\tilde{T}$ be a complete enrichment of $T$ in a language $\tilde{L}$. Assume that there exists $M \models \tilde{T}$ such that $M \vert L$ is uniformly stably embedded in every elementary extension.

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Proposition

Let $T$ be some $\mathcal{L}$-theory that eliminates imaginaries, $f$ be new function symbol and $T' \supseteq T$ be a complete $\mathcal{L} \cup \{f\}$-theory. Assume that:

1. $\Delta^!$ is injective.
2. For every $\mathcal{L}'$-definable set $X$ there exists an $\mathcal{L}$-type $p$ which is consistent with $X$.
3. There exists $M \models \bar{T}$ such that $M \mid L$ is uniformly stably embedded in every elementary extension.

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Let $T$ be some $\mathcal{L}$-theory that eliminates imaginaries, $f$ be new function symbol and $T' \supseteq T$ be a complete $\mathcal{L} \cup \{f\}$-theory. Assume that:

1. $\nabla_\omega$ is injective.
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3. There exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.

Then $T'$ eliminates imaginaries.
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▸ All the imaginaries in $\text{DCF}_0$ come from $\text{ACF}$ (and hence they can be eliminated).

▸ All the imaginaries from separably closed fields (beit with $\mu$-functions or Hassederivations) come from $\text{ACF}$.

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Thanks!