# Natural Logic

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# This talk deals with new logical systems tuned to natural language

- The raison d'être of logic is the study of inference in language.

- However, modern logic was developed in connection with the foundations of mathematics.

- So we have a mismatch, leading to
  — neglect of language in the first place
  — use of first-order logic and no other tools

- First-order logic is both too big and too small:
  — cannot handle many interesting phenomena
  — is undecidable

### Program

Show that significant parts of natural language inference can be carried out in decidable logical systems.

Whenever possible, to obtain complete axiomatizations, because the resulting logical systems are likely to be interesting.

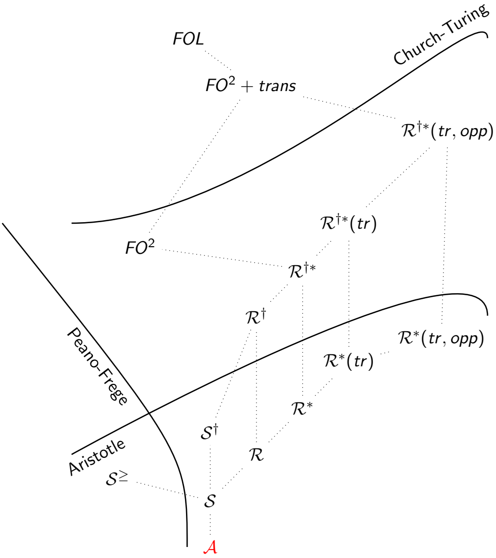
To be completely mathematical and hence to work using all tools and to make connections to fields like complexity theory, (finite) model theory, decidable fragments of first-order logic, and algebraic logic.

# Natural Logic: parallel studies

## I won't have much to say on these, but you can ask me about them

- History of logic: reconstruction of original ideas
- Philosophy of language: proof-theoretic semantics
- Philosophy of logic: why variables?
- Cognitive science: models of human reasoning
- Linguistic semantics:
  Are deep structures necessary, or can we just
  use surface forms?
  And is a complete logic a semantics?
- Computational linguistics/artificial intelligence:
  many precursors

# The Map




first-order logic

$FO^2$ + "$R$ is trans"

2 variable FO logic

† adds full $N$-negation

$\mathcal{R}^*(tr)$ + opposites

$\mathcal{R}^*$ + (transitive) comparative adjs

$\mathcal{R}$ + relative clauses

$\mathcal{S}$ + full $N$-negation

$\mathcal{R}$ = relational syllogistic

$\mathcal{S}^{\geq}$ adds $|p| \geq |q|$

$\mathcal{S}$: all/some/no p are q

$\mathcal{A}$: all p are q

# The simplest fragment "of all"

Syntax: Start with a collection of nouns.
Then the sentences are the expressions

$$\textit{All } p \textit{ are } q$$

Semantics: A model $\mathcal{M}$ is a set $M$,
together with an interpretation $[\![p]\!] \subseteq M$ for each noun $p$.

$$\mathcal{M} \models \textit{All } p \textit{ are } q \qquad \text{iff} \qquad [\![p]\!] \subseteq [\![q]\!]$$

# The semantics is trivial, as it should be

Let $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
Let $[\![a]\!] = \{1, 2, 3, 4, 5, 6\}$.
Let $[\![x]\!] = \{1, 4\}$.
Let $[\![y]\!] = \{2, 4\}$.

---

$$\mathcal{M} \models \text{All x are a}$$
$$\mathcal{M} \not\models \text{All a are x}$$
$$\mathcal{M} \not\models \text{All y are x}$$
$$\mathcal{M} \models \text{All y are a}$$
$$\mathcal{M} \models \text{All a are a}$$

# Semantic and proof-theoretic notions

If $\Gamma$ is a set of sentences, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

$\Gamma \models \varphi$ means that every $\mathcal{M} \models \Gamma$ also has $\mathcal{M} \models \varphi$.

---

All of this is semantic.

The rules are

$$\frac{}{\textit{All } p \textit{ are } p} \qquad\qquad \frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q}$$

---

A proof tree over $\Gamma$ is a finite tree $\mathcal{T}$
whose nodes are labeled with sentences,
and each node is either an element of $\Gamma$,
or comes from its parent(s) by an application of one of the rules.

$\Gamma \vdash \varphi$ means that there is a proof tree $\mathcal{T}$ for over $\Gamma$
whose root is labeled $\varphi$.

Let $\Gamma$ be the set

$$\{\textit{All a are b}, \textit{All q are a}, \textit{All b are d}, \textit{All c are d}, \textit{All a are q}\}$$

Let $\varphi$ be *All q are d*.

Here is a proof tree showing that $\Gamma \vdash \varphi$:

$$\dfrac{\textit{All q are a} \qquad \dfrac{\textit{All a are b} \qquad \textit{All b are d}}{\textit{All a are d}}\,\mathrm{B}}{\textit{All q are d}}\,\mathrm{B}$$

Suppose that $\Gamma \models$ *All p are q*.

Build a model $\mathcal{M}$, taking $M$ to be the set of variables.

Define $u \leq v$ to mean that $\Gamma \vdash$ *All u are v*.
The semantics is $[\![u]\!] = \downarrow u$.
Then $\mathcal{M} \models \Gamma$.
Hence for the $p$ and $q$ in our statement, $[\![p]\!] \subseteq [\![q]\!]$.

But by reflexivity, $p \in [\![p]\!]$.
And so $p \in [\![q]\!]$; this means that $p \leq q$.

But this is exactly what we want:
$\Gamma \vdash$ *All p are q*.

# Syllogistic Logic of *All* and *Some*

Syntax: *All p are q*, *Some p are q*

Semantics: A model $\mathcal{M}$ is a set $M$,
and for each noun p we have an interpretation $[\![p]\!] \subseteq M$.

$$\begin{array}{lll} \mathcal{M} \models \textit{All p are q} & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \textit{Some p are q} & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \emptyset \end{array}$$

Proof system:

$$\frac{}{\textit{All p are p}} \qquad \frac{\textit{All p are n} \quad \textit{All n are q}}{\textit{All p are q}}$$

$$\frac{\textit{Some p are q}}{\textit{Some q are p}} \qquad \frac{\textit{Some p are q}}{\textit{Some p are p}} \qquad \frac{\textit{All q are n} \quad \textit{Some p are q}}{\textit{Some p are n}}$$

# Example

## If there is an $n$, and if all $n$ are $p$ and also $q$, then some $p$ are $q$.

*Some n are n*, *All n are p*, *All n are q* $\vdash$ *Some p are q*.

The proof tree is

$$\dfrac{\textit{All n are q} \qquad \dfrac{\dfrac{\textit{All n are p} \quad \textit{Some n are n}}{\textit{Some n are p}}}{\textit{Some p are n}}}{\textit{Some p are q}}$$

# The languages $\mathcal{S}$ and $\mathcal{S}^\dagger$ add noun-level negation

Let us add complemented atoms $\overline{p}$ on top of
the language of All and Some,
with interpretation via set complement: $[\![\overline{p}]\!] = M \setminus [\![p]\!]$.

So we have

$$\left.\begin{array}{ll} \mathcal{S} & \left\{\begin{array}{l} \textit{All } p \textit{ are } q \\ \textit{Some } p \textit{ are } q \\ \textit{All } p \textit{ are } \overline{q} \equiv \textit{No } p \textit{ are } q \\ \textit{Some } p \textit{ are } \overline{q} \equiv \textit{Some } p \textit{ aren't } q \end{array}\right. \\ & \\ & \quad \textit{Some non-p are non-q} \end{array}\right\} \mathcal{S}^\dagger$$

# The logical system for $\mathcal{S}^\dagger$

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p}$$

$$\frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q} \qquad \frac{\textit{All } n \textit{ are } p \quad \textit{Some } n \textit{ are } q}{\textit{Some } p \textit{ are } q}$$

---

$$\frac{\textit{All } q \textit{ are } \overline{q}}{\textit{All } q \textit{ are } p}\ \textit{Zero} \qquad \frac{\textit{All } \overline{q} \textit{ are } q}{\textit{All } p \textit{ are } q}\ \textit{One}$$

$$\frac{\textit{All } p \textit{ are } \overline{q}}{\textit{All } q \textit{ are } \overline{p}}\ \textit{Antitone} \qquad \frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi}\ \textit{Ex falso quodlibet}$$

The system uses

$$\frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi} \textit{ Ex falso quodlibet}$$

and this is prima facie weaker than reductio ad absurdum.

---

One of the logical issues in this work is to determine exactly where various principles are needed.

# Completeness via representation of orthoposets

**Definition**

An orthoposet is a tuple $(P, \leq, 0, ')$ such that

| | |
|---|---|
| POSET | $\leq$ is a reflexive, transitive, and antisymmetric relation on the set $P$. |
| ZERO | $0 \leq p$ for all $p \in P$. |
| ANTITONE | If $x \leq y$, then $y' \leq x'$. |
| INVOLUTIVE | $x'' = x$. |
| INCONSISTENCY | If $x \leq y$ and $x \leq y'$, then $x = 0$. |

# Completeness via representation of orthoposets

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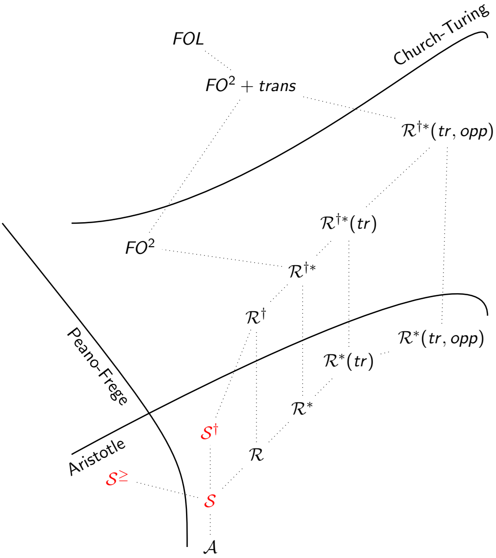
## The idea

$$\frac{\text{boolean algebra}}{\text{propositional logic}} = \frac{\text{orthoposet}}{\text{logic of All, Some and }'}$$

Completeness goes via representation.

# The Map

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# What are the simplest kinds of quantity reasoning?

Our candidate would be combinations of

*All x are y*
*Some x are y*
*No x are y*

---

*There are at least as many x as y*
*There are more x than y*
*There are at most as many x as y*
*There are fewer x than y*
*There are as many x as y*
*Most x are y*

# What are the simplest kinds of quantity reasoning?

Our candidate would be combinations of

*All x are y*
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*There are at most as many x as y*
*There are fewer x than y*
*There are as many x as y*
*Most x are y*

To add to the complexity, we could add the ability to use non-x or even to take unions and intersections.

# How can we do logic with these?

Semantics: A model $\mathcal{M}$ is a finite set $M$,
together with an interpretation $[\![p]\!] \subseteq M$ for each noun $p$.

| | | |
|---|---|---|
| $\mathcal{M} \models$ *All p are q* | iff | $[\![p]\!] \subseteq [\![q]\!]$ |
| $\mathcal{M} \models$ *Some p are q* | iff | $[\![p]\!] \cap [\![q]\!] \neq \emptyset$ |
| $\mathcal{M} \models$ *No p are q* | iff | $[\![p]\!] \cap [\![q]\!] = \emptyset$ |
| $\mathcal{M} \models$ *There are at least as many p as q* | iff | $\lvert[\![p]\!]\rvert \geq \lvert[\![q]\!]\rvert$ |
| $\mathcal{M} \models$ *There are more p than q* | iff | $\lvert[\![p]\!]\rvert > \lvert[\![q]\!]\rvert$ |
| $\mathcal{M} \models$ *There are at most as many p as q* | iff | $\lvert[\![p]\!]\rvert \leq \lvert[\![q]\!]\rvert$ |
| $\mathcal{M} \models$ *There are fewer p than q* | iff | $\lvert[\![p]\!]\rvert < \lvert[\![q]\!]\rvert$ |
| $\mathcal{M} \models$ *There are as many p as q* | iff | $\lvert[\![p]\!]\rvert = \lvert[\![q]\!]\rvert$ |
| $\mathcal{M} \models$ *Most p are q* | iff | $\lvert[\![p]\!] \cap [\![q]\!]\rvert > \frac{1}{2}\lvert[\![p]\!]\rvert$ |

# All + Some + "there are at least as many" + "there are more than"

*There are at least as many x as y* is written $\exists^{\geq}(x,y)$
*There are more x than y* is written $\exists^{>}(x,y)$

$$\frac{}{\forall(p,p)}\ (\text{AXIOM}) \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ (\text{BARBARA})$$

$$\frac{\exists(p,q)}{\exists(p,p)}\ (\text{SOME}) \qquad \frac{\exists(q,p)}{\exists(p,q)}\ (\text{CONVERSION})$$

$$\frac{\exists(p,n) \quad \forall(n,q)}{\exists(p,q)}\ (\text{DARII}) \qquad \frac{\forall(p,q) \quad \exists^{\geq}(p,q)}{\forall(q,p)}\ (\text{CARD-MIX})$$

$$\frac{\forall(p,q)}{\exists^{\geq}(q,p)}\ (\text{SUBSET-SIZE}) \qquad \frac{\exists^{\geq}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{\geq}(n,q)}\ (\text{CARD-TRANS})$$

$$\frac{\exists(p,p) \quad \exists^{\geq}(q,p)}{\exists(q,q)}\ (\text{CARD-}\exists) \qquad \frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ (\text{MORE-AT LEAST})$$

$$\frac{\exists^{>}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ (\text{MORE-LEFT}) \qquad \frac{\exists^{\geq}(n,p) \quad \exists^{>}(p,q)}{\exists^{>}(n,q)}\ (\text{MORE-RIGHT})$$

$$\frac{\exists^{\geq}(p,q) \quad \exists^{\geq}(q,p)}{\varphi}\ (\text{X})$$

# All + Some + "there are at least as many" + "there are more than"

*There are at least as many x as y* is written $\exists^{\geq}(x, y)$
*There are more x than y* is written $\exists^{>}(x, y)$

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$$\frac{\exists(p,p) \quad \exists^{\geq}(q,p)}{\exists(q,q)}\ (\text{CARD-}\exists) \qquad \frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ (\text{MORE-AT LEAST})$$

$$\frac{\exists^{>}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ (\text{MORE-LEFT}) \qquad \frac{\exists^{\geq}(n,p) \quad \exists^{>}(p,q)}{\exists^{>}(n,q)}\ (\text{MORE-RIGHT})$$

$$\frac{\exists^{\geq}(p,q) \quad \exists^{\geq}(q,p)}{\varphi}\ (\text{X})$$

**Soundness/Completeness Theorem**

$\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$.
Moreover, there's an easy algorithm to tell whether or not $\Gamma \vdash \varphi$

# Rules of inference using complemented variables $\overline{p}$

$$\frac{\forall(p,\overline{p})}{\forall(p,q)}\ (\textsc{zero}) \qquad \frac{\forall(\overline{p},p)}{\forall(q,p)}\ (\textsc{one})$$

$$\frac{\forall(q,p) \quad \exists(p,\overline{q})}{\exists^{>}(p,q)}\ (\textsc{more}) \qquad \frac{\exists^{>}(p,q)}{\exists(p,\overline{q})}\ (\textsc{more-some})$$

$$\frac{\exists^{>}(q,p)}{\exists^{>}(\overline{p},\overline{q})}\ (\textsc{more-anti}) \qquad \frac{\forall(p,q)}{\forall(\overline{q},\overline{p})}\ (\textsc{anti})$$

$$\frac{\exists^{\geq}(p,q)}{\exists^{\geq}(\overline{q},\overline{p})}\ (\textsc{card-anti}) \qquad \frac{\exists(p,p) \quad \exists^{\geq}(q,\overline{q})}{\exists(q,q)}\ (\textsc{int})$$

$$\frac{\exists^{\geq}(p,\overline{p}) \quad \exists^{\geq}(\overline{q},q)}{\exists^{\geq}(p,q)}\ (\textsc{half}) \qquad \frac{\exists^{>}(p,\overline{p}) \quad \exists^{\geq}(\overline{q},q)}{\exists^{>}(p,q)}\ (\textsc{strict half})$$

$$\frac{\exists^{\geq}(p,\overline{p}) \quad \exists^{\geq}(q,\overline{q}) \quad \exists(\overline{p},\overline{q})}{\exists(p,q)}\ (\textsc{maj}) \qquad \frac{\exists(p,q) \quad \forall(q,\overline{q})}{\varphi}\ (\mathrm{X})$$

# Rules, rules, rules

$$\frac{}{\forall(p,p)}\ (\text{axiom}) \qquad \frac{\forall(n,p) \quad \forall(p,q)}{\forall(n,q)}\ (\text{Barbara}) \qquad \frac{\exists(p,q)}{\exists(p,p)}\ (\text{some})$$

$$\frac{\exists(q,p)}{\exists(p,q)}\ (\text{conversion}) \qquad \frac{\forall(p,q)}{\forall(q',p')}\ (\text{anti}) \qquad \frac{\forall(p,p')}{\forall(p,q)}\ (\text{zero})$$

$$\frac{\exists(p,n) \quad \forall(n,q)}{\exists(p,q)}\ (\text{Darii}) \qquad \frac{\forall(p',p)}{\forall(q,p)}\ (\text{one}) \qquad \frac{\forall(p,q)}{\exists^{\geq}(q,p)}\ (\text{subset-size})$$

$$\frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ (\text{card-mon}) \qquad \frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ (\text{card-anti}) \qquad \frac{\forall(p,q) \quad \exists^{\geq}(p,q)}{\forall(q,p)}\ (\text{card-mix})$$

$$\frac{\exists(p,p) \quad \exists^{\geq}(p,q)}{\exists(q,q)}\ (\text{card-}\exists) \qquad \frac{\forall(q,p) \quad \exists(p,q')}{\exists^{>}(p,q)}\ (\text{more}) \qquad \frac{\exists^{>}(p,q)}{\exists(p,q')}\ (\text{more-some})$$

$$\frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ (\text{more-at least}) \qquad \frac{\exists^{>}(n,p) \quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ (\text{more-left}) \qquad \frac{\exists^{>}(q,p)}{\exists^{>}(p',q')}\ (\text{more-anti})$$

$$\frac{\exists(p,p) \quad \exists^{\geq}(q,q')}{\exists(q,q)}\ (\text{int}) \qquad \frac{\exists^{\geq}(p,p') \quad \exists^{\geq}(q',q)}{\exists^{\geq}(p,q)}\ (\text{half}) \qquad \frac{\exists^{>}(p,p') \quad \exists^{\geq}(q',q)}{\exists^{>}(p,q)}\ (\text{strict half})$$

$$\frac{\exists^{\geq}(p,p') \quad \exists^{\geq}(q,q') \quad \exists(p',q')}{\exists(p,q)}\ (\text{maj})$$

$$\frac{\exists(p,q) \quad \forall(p,q')}{\varphi}\ (\text{X}) \qquad \frac{\exists^{>}(p,q) \quad \exists^{\geq}(q,p)}{\varphi}\ (\text{X})$$

The logic has been implemented in Sage, and the implementation is currently available on `https://cloud.sagemath.com`.

(That is, I can share it.)

For example, one may enter

```
assumptions= ['All non-a are b',
'There are more c than non-b',
'There are more non-c than non-b',
'There are at least as many non-d as d',
'There are at least as many c as non-c',
'There are at least as many non-d as non-a']
conclusion = 'All a are non-c'
follows(assumptions,conclusion)
```

We get

```
The conclusion does not follow
Here is a counter-model.
We take the universe of the model to be
{0, 1, 2, 3, 4, 5}
noun semantics
a      {2, 3}
b      {0, 1, 4, 5}
c      {0, 2, 3}
d      {}
```

So it gives the semantics of `a`, `b`, `c`, and `d` as subsets of $\{0, \ldots, 5\}$. Notice that the assumptions are true in the model, but the conclusion is false.

Here is an example of a derivation found by our implementation. We ask whether the putative conclusion below really follows:

All non-x are x
Some non-y are z
―――――――――――
There are more x than y

```
Here is a formal proof in our system:
1 All non-x are x      Assumption
2 All y are x     One 1
3 All non-x are x      Assumption
4 All non-y are x      One 3
5 Some non-y are z     Assumption
6 Some non-y are non-y     Some 5
7 Some non-y are x     Darii 4 6
8 Some x are non-y     Conversion 7
9 There are more x than y     More 2 8
```

$$\frac{}{\text{All } X \text{ are } X} \qquad \frac{\text{All } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{All } X \text{ are } Z}$$

$$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Some } X \text{ are } Z}$$

Can you think of any valid laws that add Most $X$ are $Y$ on top of All $X$ are $Y$ and Some $X$ are $Y$?

## THIS TALK: ALL, SOME, MOST

$$\frac{}{\text{All } X \text{ are } X} \qquad \frac{\text{All } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{All } X \text{ are } Z}$$

$$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Some } X \text{ are } Z}$$

---

$$\frac{\text{Most } X \text{ are } Y}{\text{Some } X \text{ are } Y}\, m_1 \qquad \frac{\text{Some } X \text{ are } X}{\text{Most } X \text{ are } X}\, m_2 \qquad \frac{\text{Most } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Most } X \text{ are } Z}\, m_3$$

$$\frac{\text{Most } X \text{ are } Z \quad \text{All } X \text{ are } Y \quad \text{All } Y \text{ are } X}{\text{Most } Y \text{ are } Z}\, m_4$$

$$\frac{\text{All } Y \text{ are } X \quad \text{All } X \text{ are } Z \quad \text{Most } Z \text{ are } Y}{\text{Most } X \text{ are } Y}\, m_5$$

$$\frac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \quad \cdots \quad X_n \triangleright_{A,B} Y_n \quad Y_n \triangleright_{B,A} X_1}{\text{Some } A \text{ are } B}\, \triangleright$$

# The last infinite batch of rules

$$\frac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \quad \cdots \quad X_n \triangleright_{A,B} Y_n \quad Y_n \triangleright_{B,A} X_1}{\text{Some } A \text{ are } B} \triangleright$$

---

Examples:

$$\frac{\text{Most } Z \text{ are } X \quad \text{Most } Z \text{ are } Y}{\text{Some } X \text{ are } Y} \triangleright$$

**You call this an inference rule?!**

From

$$\begin{array}{l} \text{Most } X \text{ are } B', \text{All } A' \text{ are } A, \text{Most } Y \text{ are } A', \text{All } B' \text{ are } B, \text{All } X \text{ are } Y \\ \text{Most } Y \text{ are } A'', \text{All } A'' \text{ are } A, \text{Most } X \text{ are } B'', \text{All } B'' \text{ are } B, \text{All } A'' \text{ are } X \end{array}$$

infer

$$\text{Some } A \text{ are } B.$$

**THEOREM (JÖRG ENDRULLIS & LM (WOLLIC 2015))**

The logical system for this language is complete.

**THEOREM**

Infinitely many axioms are needed in the system.

**THEOREM**

The decision problem for the consequence relation

$$\Gamma \vdash \varphi$$

is in polynomial time.

With Tri Lai (then a grad student at IU in combinatorics) we showed that

- Most $X$ are $Y$
- boolean connectives, especially negation

has a very simple proof system and is also in PTIME.

- Get a such complete logic for

| | | |
|---|---|---|
| All $X$ are $Y$ | Some $X$ are $Y$ | Most $X$ are $Y$ |
| No $X$ are $Y$ | $\exists^{\geq}(X, Y)$ | |

  and sentential $\wedge$, $\vee$, and $\neg$.

- Alternatively, prove that there is no such logic.
- Investigate the algorithmic properties of the logic.

# Inference with relative clauses

What do you think about this one?

$$\frac{\textit{All skunks are mammals}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

It follows, using an interesting antitonicity principle:

$$\frac{\textit{All skunks are mammals}}{\textit{All who respect all mammals respect all skunks}}$$

It follows, using an interesting antitonicity principle:

$$\cfrac{\cfrac{\textit{All skunks are mammals}}{\textit{All who respect all mammals respect all skunks}}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

# All + Verbs + Relative Clauses

We start with two sets:

- a set of nouns.
- a set of verbs.

We make terms as follows:

- If x is a noun, then x is a term.
- If r is verb and x is a term, then r all x is a term.

We make sentences as follows:

- If x and y are terms, then

  All x y

  is a sentence.

Let's say

- $\mathbf{P} = \{$dogs, cats, birds, ants, $\ldots\}$
- $\mathbf{R} = \{$see, like, hate, fear, respect, $\ldots\}$

Here are some terms of $\mathcal{A}(\mathcal{RC})$:

- dogs
- see all dogs
- respect all (see all dogs)
- love all (respect all (see all dogs))

Note that there are infinitely many terms, and terms may occur in terms.

Let's say

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Here are some terms of $\mathcal{A}(\mathcal{RC})$:

- dogs
- see all dogs
- respect all (see all dogs)
  read as respect all who see all dogs
- love all (respect all (see all dogs))
  read as love all who respect all who see all dogs

Note that there are infinitely many terms, and terms may occur in terms.

We read these in English using relative clauses.

We make proof trees using the following rules

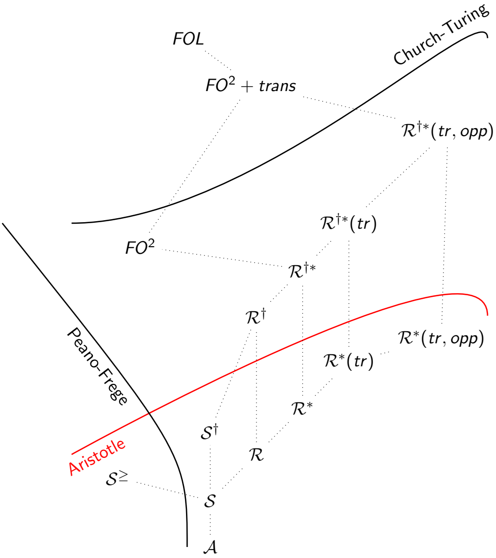
$$\frac{}{\mathsf{All}\ x\ x}\ \textsc{Axiom} \qquad \frac{\mathsf{All}\ x\ y \quad \mathsf{All}\ y\ z}{\mathsf{All}\ x\ z}\ \textsc{Barbara}$$

$$\frac{\mathsf{All}\ y\ x}{\mathsf{All}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ y)}\ \textsc{Anti}$$

Note that we are using this with $x$, $y$, and $z$ as terms, not only as unary variables.

$$\dfrac{\dfrac{\text{All skunks mammals}}{\text{All (love all mammals) (love all skunks)}}\ \textsc{anti}}{\text{All (hate all (love all skunks)) (hate all (love all mammals))}}\ \textsc{anti}$$

# The Map




first-order logic

$FO^2$ + "$R$ is trans"

2 variable FO logic

† adds full $N$-negation

$\mathcal{R}^*(tr)$ + opposites

$\mathcal{R}^*$ + (transitive)
comparative adjs

$\mathcal{R}$ + relative clauses

$\mathcal{S}$ + full $N$-negation

$\mathcal{R}$ = relational syllogistic

$\mathcal{S}^{\geq}$ adds $|p| \geq |q|$

$\mathcal{S}$: all/some/no p are q

$\mathcal{A}$: all p are q

$\mathcal{R}^{\dagger}$ and $\mathcal{R}^{\dagger *}$ lie beyond the Aristotle boundary, due to full negation on nouns.

It is possible to formulate a logical system with
a restricted notion of variables,
prove completeness,
and yet stay inside the Turing boundary.

It's a fairly involved definition, so I've hidden the details to slides after the end of the talk.

Instead, I'll show examples.

# Example of a proof in the system

## From all keys are old items, infer everyone who owns a key owns an old item

$$
\dfrac{[\exists(key, own)(x)]^2 \qquad \dfrac{[own(x,y)]^1 \qquad \dfrac{[key(y)]^1 \quad \forall(key, old\text{–}item)}{old\text{–}item(y)}\ \forall E}{\exists(old\text{–}item, own)(x)}\ \exists I}{\dfrac{\exists(old\text{–}item, own)(x)}{\forall(\exists(key, own), \exists(old\text{–}item, own))}\ \forall I^2}\ \exists E^1
$$

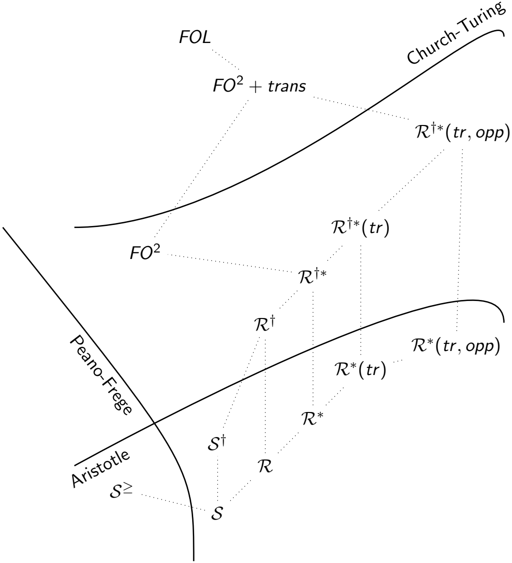
# Example of a proof in the system

## From all keys are old items, infer everyone who owns a key owns an old item

| | | |
|---|---|---|
| 1 | $\forall(key, old\text{–}item)$ | hyp |
| 2 | $\quad \exists(key, own)(x)$ | hyp |
| 3 | $\quad key(y)$ | $\exists E$, 2 |
| 4 | $\quad own(x, y)$ | $\exists E$, 2 |
| 5 | $\quad old\text{–}item(y)$ | $\forall E$, 1, 3 |
| 6 | $\quad \exists(old\text{–}item, own)(x)$ | $\exists I$, 4, 5 |
| 7 | $\forall(\exists(key, own), \exists(old\text{–}item, own))$ | $\forall I$, 1–6 |

| | | | |
|---|---|---|---|
| 1 | | John is a man | Hyp |
| 2 | | Any woman is a mystery to any man | Hyp |
| 3 | Jane | Jane is a woman | Hyp |
| 4 | | Any woman is a mystery to any man | R, 2 |
| 5 | | Jane is a mystery to any man | Any Elim, 4 |
| 6 | | John is a man | R, 1 |
| 7 | | Jane is a mystery to John | Any Elim, 6 |
| 8 | | Any woman is a mystery to John | Any intro, 3, 7 |

# Review

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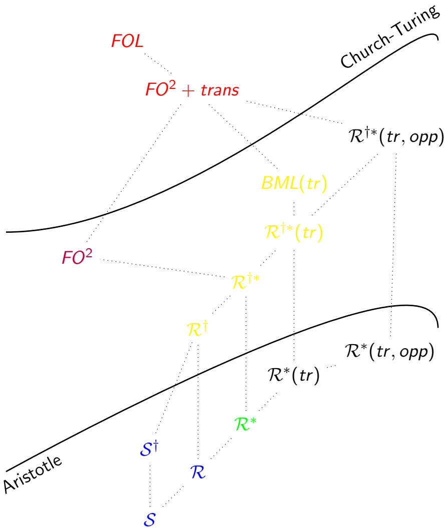
$\mathcal{R}$ + relative clauses
$\mathcal{S}$ + full $N$-negation
$\mathcal{R}$ = relational syllogistic
$\mathcal{S}^{\geq}$ adds $|p| \geq |q|$
$\mathcal{S}$: all/some/no p are q

# Complexity

## (mostly) best possible results on the validity problem




undecidable
Church 1936
Grädel, Otto, Rosen 1999

in co-NEXPTIME
EXPTIME
Lutz & Sattler 2001

Co-NEXPTIME
Grädel, Kolaitis, Vardi '97
EXPTIME
Pratt-Hartmann 2004

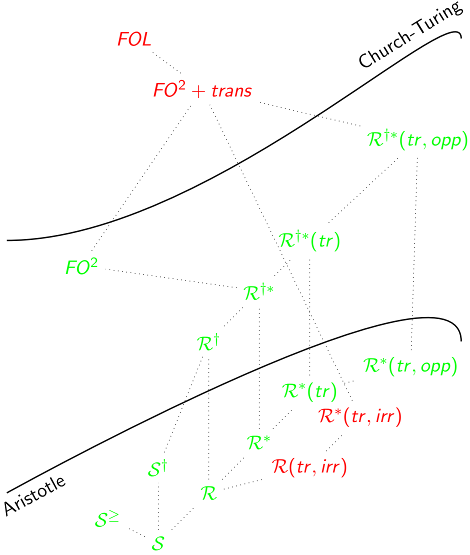
lower bounds also open

Co-NP
McAllester & Givan 1992

NLOGSPACE

| | | |
|---|---|---|
| $\mathcal{S}$ | NLOGSPACE | lower bound via reachability problem for directed graphs |
| $\mathcal{S}^{\dagger}$ | NLOGSPACE | upper bound via $2\mathrm{SAT}$ |
| $\mathcal{R}$ | NLOGSPACE | upper bound takes special work based on the proof system |
| $\mathcal{R}^{\dagger}$ | EXPTIME | lower bound via $K^U$, Hemaspaandra 1996 |
| $\mathcal{R}^{*\dagger}$ | EXPTIME | upper bound by Pratt-Hartmann 2004 |
| $BML(tr)$ | EXPTIME | Boolean modal logic on transitive models Lutz and Sattler 2001 |
| $\mathcal{R}^{*}$ | Co-NPTIME | essentially in McAllester and Givan 1992 |
| FO$^2$ | NEXPTIME | Grädel, Kolaitis, and Vardi 1997 |

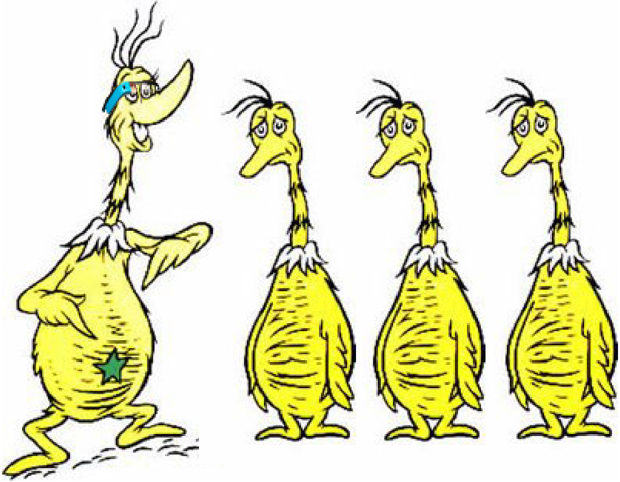
# The finite model property: Yes$^{\downarrow}$ and No$^{\uparrow}$

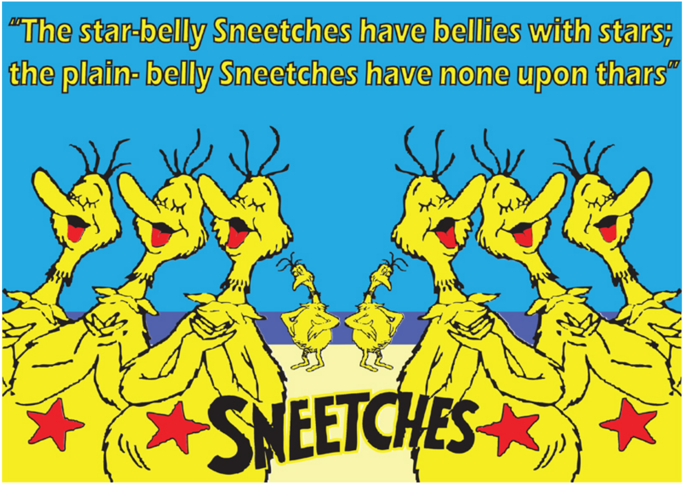


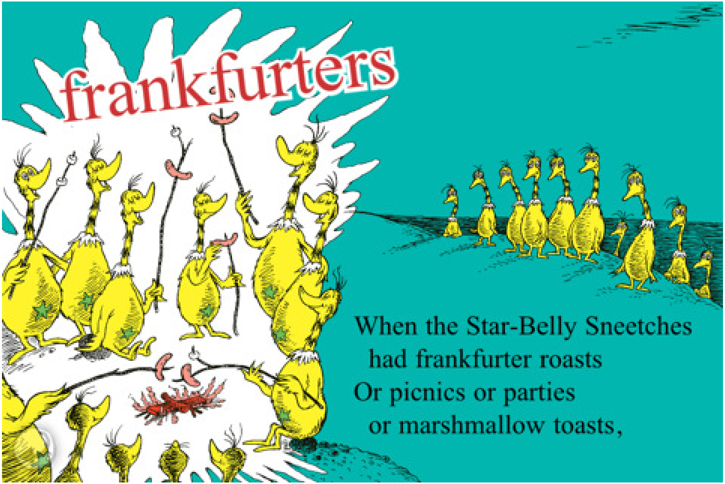

filtration of a
Henkin model

Mortimer 1975

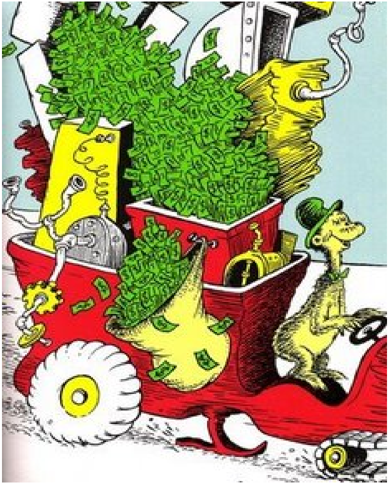
*irr* means that
comparative adjectives
must have irreflexive
interpretations.
$\forall(p, \exists(p, r)) + \exists p$




“The star-belly Sneetches have bellies with stars;
the plain- belly Sneetches have none upon thars”
SNEETCHES




When the Star-Belly Sneetches
had frankfurter roasts
Or picnics or parties
or marshmallow toasts,



Sylvester McMonkey McBean said, “you can’t teach a Sneetch”

How can a person or computer
answers questions involving a word which they don't know?

A word like Sneetch.

How can a person or computer
answers questions involving a word which they don't know?

A word like Sneetch.

**What "follows from" means**

One sentence follows from a second sentence
if every time we use the first sentence in a true way,
we could also have used the second.

How can a person or computer
answers questions involving a word which they don't know?

A word like Sneetch.

**WHAT "FOLLOWS FROM" MEANS**

One sentence follows from a second sentence
if every time we use the first sentence in a true way,
we could also have used the second.

If we say

every animal hops

then it follows that

every Sneetch moves

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

---

Let's talk about a situation where

all Sneetches dance.

Which one would be true?

- all Star-Belly Sneetches dance
- all animals dance

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

---

- all Star-Belly Sneetches dance true
- all animals dance false

We write

all Sneetches$^{\downarrow}$ dance

$\uparrow$

animal
Sneetch
Star-Belly Sneetch

$\uparrow$

move
dance
waltz

---

all Sneetches$^{\downarrow}$ dance

What arrow goes on "dance"?

- all Sneetches waltz
- all Sneetches move

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

---

We write

$$\text{all Sneetches}^{\downarrow}\ \text{dance}^{\uparrow}$$

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

---

Let's put the arrows on the words Sneetches and dance.

1. No Sneetches dance.
2. If you play loud enough music, any Sneetch will dance.
3. Any Sneetch in Zargonia would prefer to live in Yabistan.
4. If any Sneetch dances, McBean will dance, too.

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

---

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move
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| | |
|---|---|
| animal | move |
| Sneetch | dance |
| Star-Belly Sneetch | waltz |

---

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$\uparrow$

animal

Sneetch

Star-Belly Sneetch

$\uparrow$

move

dance

waltz

---

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# What goes up? What goes down?

$$f(x, y) = y - x \tag{1}$$

$$g(x, y) = x + \frac{2}{y} \tag{2}$$

$$h(v, w, x, y, z) = \frac{x - y}{2^{z-(v+w)}} \tag{3}$$

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$$f(x^{\downarrow}, y^{\uparrow}) = y - x \tag{1}$$

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$$h(v^{\uparrow}, w^{\uparrow}, x^{\uparrow}, y^{\downarrow}, z^{\downarrow}) = \frac{x - y}{2^{z-(v+w)}} \tag{3}$$

The $^{\uparrow}$ and $^{\downarrow}$ notations have the same meaning in language as in math!

This is not an accident!

$$3 \times \frac{2}{3} = ?$$

$$\frac{7}{4} \times 4 = ?$$

$$\cancel{3} \times \frac{2}{\cancel{3}} = 2$$

$$\frac{7}{\cancel{4}} \times \cancel{4} = 7$$

You can cancel on the left.
You can cancel on the right.

$$\frac{7\cdot\cancel{5}\cdot 3}{8\cdot\cancel{5}\cdot 2} = \frac{21}{40}$$

$$\frac{\cancel{8}\cdot 5\cdot 3}{7\cdot 5\cdot\cancel{8}} = \frac{15}{40}$$

You can cancel down the middle.
You can cancel end-to-end.

$$\frac{7\cdot\cancel{5}\cdot 3}{8\cdot\cancel{5}\cdot 2} = \frac{21}{40}$$

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You can cancel down the middle.
You can cancel end-to-end.

$$\frac{7\cdot\cancel{4}\cdot 5}{\cancel{4}\cdot\cancel{4}\cdot 2} = \frac{35}{2}$$

But if you cancel wrongly, ...

$\backslash$ means "look left"
$/$ means "look right"

$$X \times (Y \backslash X) = Y$$

$$(X/Y) \times Y = X$$

# Categorial grammar

$$\dfrac{\text{McBean: } NP \qquad \dfrac{\text{teased: } (S\backslash NP)/NP \qquad \dfrac{\text{a : } NP/N \quad \text{Sneetch : N}}{\text{a Sneetch: } NP}}{\text{teased a Sneetch: } S\backslash NP}}{\text{McBean teased a Sneetch: } S}$$

$$\dfrac{\text{Seuss: } NP \qquad \dfrac{\dfrac{\text{criticized: } (S\backslash NP)/NP \quad \text{McBean: } NP}{\text{criticized McBean: } S\backslash NP} \qquad \text{gently: } (S\backslash NP)\backslash(S\backslash NP)}{\text{criticized McBean gently: } S\backslash NP}}{\text{Seuss criticized McBean gently: } S}$$

| syntactic category | name in traditional grammar |
|---|---|
| $S$ | sentence |
| N | noun |
| $NP$ | noun phrase |
| N/N | adjective |
| $S\backslash NP$ | verb phrase |
| $S\backslash NP$ | intransitive verb |
| $(S\backslash NP)\backslash(S\backslash NP)$ | adverb |
| $(S\backslash NP)/NP$ | transitive verb |
| $NP$/N | determiner |
| $(\mathrm{N}\backslash\mathrm{N})/(S\backslash NP)$ | relative pronoun |

# Proposal: marry grammar and inference

# Proposal:
# marry grammar and inference

$$\dfrac{\dfrac{\text{Sneetch} \leq \text{animal}}{\text{every animal}^{\downarrow} \leq \text{every Sneetch}^{\downarrow}} \qquad \dfrac{\text{every} \leq \text{most}}{\text{every Sneetch}^{\downarrow} \leq \text{most Sneetches}^{\downarrow}}}{\dfrac{\text{every animal}^{\downarrow} \leq \text{most Sneetches}^{\downarrow}}{\text{every animal}^{\downarrow}\ \text{hops}^{\uparrow} \leq \text{most Sneetches}^{\downarrow}\ \text{move}^{\uparrow}}}$$

This is how a computer could do the reasoning:

if every animal hops
then most Sneetches move

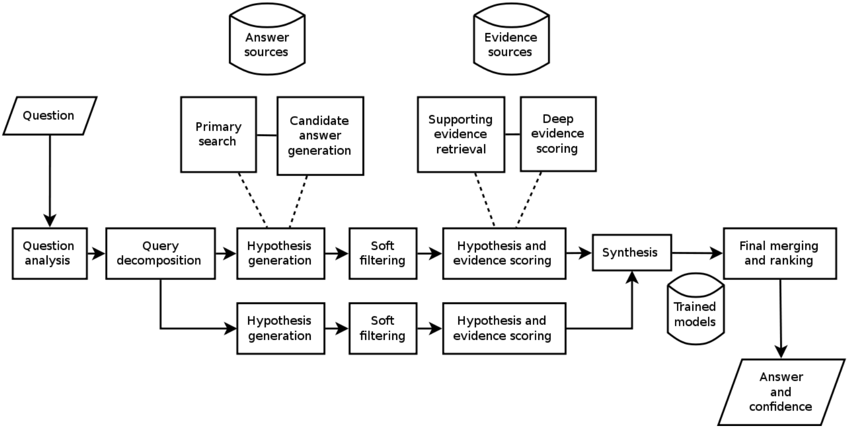
- linguistics
- logic
- artificial intelligence/cognitive science
- mathematics
- philosophy

- linguistics
- logic
- artificial intelligence/cognitive science
- mathematics
- philosophy

Natural Sciences

Humanities

# What about Watson?



# Natural logic: what I hope to have gotten across

## Program

Show that significant parts of natural language inference can be carried out in decidable logical systems.

Whenever possible, to obtain complete axiomatizations, because the resulting logical systems are likely to be interesting.

To be completely mathematical and hence to work using all tools and to make connections to fields like complexity theory, (finite) model theory, decidable fragments of first-order logic, and algebraic logic.

## Last words for logicians

- We must ask whether a complete proof system is a semantics.
- We should not be afraid of doing logic beyond logic.
- Joining the perspectives of semantics, complexity theory, proof theory, cognitive science, and computational linguistics should allow us to ask interesting questions and answer them.

# Influences

- Aristotle
- Boole, de Morgan
- (1960's and '70's) Montague, Fitch, Lakoff
- McAllester and Givan, Nishihara, Morita, Iwata, etc.
- Sommers, Corcoran, Martin
- van Benthem
- Ian Pratt-Hartmann and Thomas Icard
- maybe you, why not?

# Living in two worlds